Structural Origins of Exponential Persistence II: Spectral Slow Manifolds and Finite Persistence Coordinates SOEP II

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Abstract

Persistence dynamics in open metastable systems are governed by slow spectral structure of effective generators. Under structural axioms introduced in SOEP-I, metastable dynamics produce a finite cluster of slow eigenmodes separated by a spectral gap from fast relaxation modes. Persistence observables are therefore controlled by finite-dimensional persistence coordinates associated with slow spectral modes. These results establish spectral dimensional reduction for persistence dynamics and provide the bridge between structural persistence inevitability and analytic persistence asymptotics.

1 Introduction

Structural analysis in SOEP-I establishes inevitability of exponential persistence scaling under openness, metastability, constraint geometry, and multiplicative survival aggregation. The present work establishes spectral mechanisms underlying persistence dynamics.

The central objective is to prove that persistence observables evolve on a finite-dimensional slow spectral manifold generated by the effective generator of coarse-grained dynamics.

2 Effective Generator Framework

2.1 State Space

Let (X, d) be a compact metric space.

2.2 Effective Dynamics

Let L_Θ denote the effective generator associated with coarse-grained stochastic dynamics on X.

2.3 Metastable Domain

Let $D \subset X$ denote a metastable domain with mixing time τ_{mix} and escape time τ_{esc} satisfying

$$\tau_{\text{mix}} \ll \tau_{\text{esc}}.$$

3 Killed Process and Dirichlet Generator

Definition 3.1 (Killed Semigroup)**.** *Define the killed semigroup*

$$P_t^D f(x) = \mathbb{E}_x \left[f(X_t) \mathbf{1}_{t<\tau} \right].$$

Definition 3.2 (Dirichlet Generator)**.** *The generator L_D associated with P_t^D is called the Dirichlet generator restricted to D.*

4 Quasi-Stationary Spectral Structure

Definition 4.1 (Quasi-Stationary Distribution)**.** *A probability distribution π_D supported on D is quasi-stationary if*

$$\mathbb{P}_{\pi_D}(X_t \in A \mid \tau > t) = \pi_D(A)$$

for all measurable $A \subset D$.

Theorem 4.2 (QSD Spectral Correspondence)**.** *There exists a quasi-stationary distribution π_D such that*

(i) π_D is proportional to the principal eigenfunction of L_D^*,

(ii) survival probability satisfies

$$\mathbb{P}_{\pi_D}(\tau > t) = e^{-\lambda_1 t},$$

where $\lambda_1 > 0$ is the principal Dirichlet eigenvalue.

Proof. The killed semigroup is positive and compact under standard smoothing assumptions. The Krein–Rutman theorem implies existence of a principal positive eigenfunction. Normalization yields the quasi-stationary distribution. Spectral expansion of the killed semigroup yields exponential survival law.

5 Principal Escape Eigenvalue

Theorem 5.1 (Escape Rate Eigenvalue Representation)**.** *The mean persistence time satisfies*

$$\mathbb{E}[\tau] \sim \lambda_1^{-1}.$$

Proof. The survival probability is dominated by the principal spectral term of the killed semigroup. Integration of exponential survival law yields the result.

6 Slow–Fast Spectral Structure

6.1 Spectrum of Effective Generator

Let $\sigma(L_\Theta)$ denote the spectrum of L_Θ.

Theorem 6.1 (Finite Slow Spectral Cluster)**.** *There exists a finite integer m such that*

$$\sigma(L_\Theta) = \{0\} \cup \{\lambda_1, \dots, \lambda_m\} \cup \sigma_{\text{fast}},$$

where

$$|\lambda_k| \ll |\Re(\lambda_{\text{fast}})|.$$

Proof Strategy. The proof follows from quasi-compactness of the semigroup generated by L_Θ. Quasi-compactness implies finite discrete spectrum outside essential spectrum. Metastability implies existence of small eigenvalues associated with slow escape modes.

Theorem 6.2 (Spectral Gap Separation)**.** *There exists $\Delta > 0$ such that*

$$\min_{j>m} |\Re(\lambda_j)| - \max_{k \le m} |\Re(\lambda_k)| \ge \Delta.$$

Proof Strategy. Fast mixing modes produce spectral bound away from zero. Escape modes produce exponentially small eigenvalues. The difference yields spectral gap.

7 Finite Persistence Coordinate Reduction

Theorem 7.1 (Finite Persistence Dimension)**.** *There exists a finite-dimensional persistence coordinate vector*

$$I = (I_1, \dots, I_m)$$

such that persistence observables depend only on I up to exponentially small corrections.

Proof Strategy. Spectral expansion of semigroup evolution implies that long-time dynamics are dominated by slow eigenmodes. Fast modes decay exponentially with mixing rate.

8 Persistence Spectral Projection

Let P_{slow} denote projection onto slow spectral subspace.

Theorem 8.1 (Persistence Projection Theorem)**.** *There exist constants $C > 0$ and $\lambda_{\text{mix}} > 0$ such that*

$$\|e^{tL_\Theta} - P_{\text{slow}} e^{tL_\Theta} P_{\text{slow}}\| \le C e^{-\lambda_{\text{mix}} t}.$$

Proof Strategy. The proof follows from Dunford contour representation of semigroup and resolvent bounds away from slow spectral cluster.

9 Spectral Stability Under Perturbations

9.1 Noise Perturbations

Consider perturbed generators of the form

$$L_\epsilon = L_\Theta + \epsilon V,$$

where V is a bounded linear operator.

Theorem 9.1 (Noise Perturbation Spectral Stability)**.** *Let $\lambda_k(0)$ denote slow eigenvalues of L_Θ and let $\lambda_k(\epsilon)$ denote eigenvalues of L_ϵ. Then for sufficiently small ϵ,*

$$|\lambda_k(\epsilon) - \lambda_k(0)| \leq C\epsilon.$$

Proof Strategy. Apply analytic perturbation theory for linear operators. The spectral gap prevents eigenvalue crossings. Standard Kato perturbation results imply continuous dependence of isolated eigenvalues.

9.2 Constraint Geometry Perturbations

Let constraint perturbations induce generator perturbations through modified domain or boundary structure.

Theorem 9.2 (Constraint Perturbation Spectral Stability)**.** *Small smooth perturbations of constraint geometry preserve slow spectral cluster structure and spectral gap separation.*

Proof Strategy. Constraint perturbations induce bounded perturbations of the generator under smooth domain deformation. Apply stability of isolated eigenvalues under bounded perturbations.

10 Multi-Domain Metastability

Assume existence of metastable domains $D_1, \ldots, D_r$.

Theorem 10.1 (Multi-Domain Persistence Vector Representation)**.** *Persistence dynamics are governed by a finite persistence coordinate vector*

$$I = (I^{(1)}, \ldots, I^{(r)}),$$

where each coordinate corresponds to slow spectral modes associated with domain escape transitions.

Proof Strategy. Potential-theoretic metastability theory implies one dominant slow mode per metastable domain. Reduced dynamics can be approximated by a finite Markov chain between metastable wells.

Theorem 10.2 (Eigenvector Localization Near Metastable Domains)**.** *Slow eigenfunctions are spatially localized near metastable domains.*

Proof Strategy. Large-deviation potential landscape implies eigenfunction concentration near local potential minima. WKB asymptotics or potential-theoretic capacity methods yield localization.

11 Structural Spectral Reduction Principle

Theorem 11.1 (Structural Spectral Reduction)**.** *Under SOEP structural axioms, persistence dynamics reduce to evolution on a finite-dimensional slow spectral manifold generated by slow eigenfunctions of the effective generator.*

Proof. Finite slow spectral cluster exists by quasi-compactness and metastability. Persistence observables depend only on slow spectral coefficients due to exponential decay of fast modes. Projection theorem ensures long-time dynamics remain confined to slow spectral manifold.

12 Physical Interpretation of Slow Spectral Structure

Slow eigenmodes correspond to escape channels between metastable domains. Fast eigenmodes correspond to internal equilibration within domains. Persistence observables therefore encode escape channel activation structure.

13 Boundary of Spectral Reduction Validity

Spectral reduction may fail under the following conditions:

- Continuous spectrum without discrete metastable separation,
- Infinite memory stochastic forcing,
- Critical systems with vanishing spectral gap,
- Deterministic Hamiltonian confinement without escape.

14 Bridge to Analytic Persistence Asymptotics (SOEP-III)

Spectral reduction implies persistence distributions are dominated by finite slow eigenvalue structure. This enables analytic asymptotic characterization using Laplace transform and saddle-point methods developed in subsequent work.

15 Conclusion

Persistence dynamics in open metastable systems are governed by finite slow spectral structure. This establishes spectral dimensional reduction of persistence observables and provides the mathematical bridge between structural inevitability and analytic asymptotic universality theory.

A Quasi-Compactness Background

Quasi-compactness results for positive semigroups imply decomposition of spectrum into discrete eigenvalues and essential spectrum separated by spectral radius gap. These results follow from compact kernel smoothing combined with exponential moment bounds.

B Perturbation Theory Background

Analytic perturbation theory for linear operators ensures stability of isolated eigenvalues under small bounded perturbations. See classical operator perturbation theory references.

C Metastable Spectral Theory Background

Potential theory and metastability theory establish correspondence between slow eigenvalues and escape rates between metastable domains.

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The conceptual framework, theoretical direction, and primary scientific contributions presented in this work originate from the author. Automated computational drafting tools were used to assist in portions of formal mathematical expression and manuscript preparation.

All theoretical decisions, structural design, and final formulations were determined and verified by the author. The author retains full intellectual ownership of the work and accepts full responsibility for its content.