Structural Origins of Exponential Persistence I: Minimal Structural Axioms and Emergent Persistence Scaling SOEP I

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Abstract

Persistence and escape times arise across physics, biology, and complex engineered systems, yet a structural explanation for the ubiquity of exponential persistence scaling remains incomplete. This work introduces a minimal structural framework for open metastable dynamical systems and demonstrates that exponential persistence emerges generically from openness, metastability, constraint geometry, and multiplicative survival aggregation. Under these axioms, effective stochastic forcing, large-deviation escape structure, and exponential survival kernels arise as necessary consequences rather than model-specific assumptions. These results establish a structural foundation for persistence scaling and provide the first step toward a broader persistence universality program.

1 Introduction

Persistence phenomena appear across multiple scientific domains, including metastable physical systems, reliability engineering, biological state switching, and complex network dynamics. In many such systems, escape times from metastable domains exhibit exponential scaling behavior. While model-specific derivations exist, a general structural explanation remains incomplete.

The objective of this work is to establish minimal structural conditions under which exponential persistence scaling necessarily emerges. The approach is axiomatic and structural rather than model-specific. The results provide the foundational layer for a broader structural persistence theory.

2 Mathematical Setting

2.1 State Space

Let (X, d) denote a compact metric space representing observable system states.

2.2 Observed Dynamics

Let $x(t)$ denote system evolution on X. The evolution may be deterministic, stochastic, or effective stochastic after coarse-graining. The evolution is assumed to admit a generator description or equivalent transition semigroup description.

2.3 Open System Embedding

Assume existence of a full system

$$X_{\text{full}} = X \times E,$$

where E represents environmental degrees of freedom. Environmental dynamics are assumed to possess finite correlation time.

2.4 Metastable Domain

Definition 2.1 (Metastable Domain). *A subset $D \subset X$ is called metastable if internal mixing time τ_{mix} satisfies*

$$\tau_{mix} \ll \tau_{esc},$$

where τ_{esc} denotes the mean escape time from D.

2.5 Persistence Time

Definition 2.2 (Persistence Time). *Persistence time is defined as the first exit time*

$$\tau = \inf\{t > 0 : x(t) \notin D\}.$$

3 Minimal Structural Axioms

Axiom 3.1 (Openness). *The observed dynamics arise as a projection of dynamics on $X_{full} = X \times E$, where environmental fluctuations possess finite correlation time.*

Axiom 3.2 (Metastability). *There exists a domain $D \subset X$ such that mixing occurs on a timescale much shorter than escape.*

Axiom 3.3 (Constraint Geometry). *Escape trajectories belong to a restricted subset of all admissible system trajectories.*

Axiom 3.4 (Survival Aggregation). *Independent constraint channels combine multiplicatively at the survival probability level.*

4 Emergence of Effective Stochastic Dynamics

Theorem 4.1 (Effective Noise Emergence). *Under the openness axiom, coarse-grained dynamics admit an effective stochastic representation.*

Proof. Let unresolved environmental forcing be denoted by $\eta(t)$. Finite correlation time implies that integrated forcing over coarse time intervals can be expressed as a sum of weakly dependent random variables. By central limit accumulation, the coarse-grained forcing converges in distribution to Gaussian noise or finite-variance stochastic forcing. This produces an effective stochastic evolution for observed variables.

5 Large-Deviation Escape Structure

Theorem 5.1 (Generic Large-Deviation Escape)**.** *Under metastability and effective stochastic forcing, escape probabilities are dominated by minimal action trajectories.*

Proof. Rare escape events correspond to atypical fluctuation paths. Under standard large-deviation principles, path probabilities satisfy

$$P[x(\cdot)] \asymp \exp\left(-\frac{S[x]}{\Theta}\right),$$

where $S[x]$ is the action functional. Dominant contributions arise from trajectories minimizing $S[x]$ subject to escape boundary conditions.

Corollary 5.2. *Escape rate scales exponentially with the minimal action barrier:*

$$\gamma \sim \exp(-C/\Theta).$$

6 Constraint-Induced Barrier Amplification

Theorem 6.1 (Constraint Action Amplification)**.** *Let $\mathcal{T}$ denote the set of all admissible escape trajectories and let $\mathcal{T}_{\text{cons}} \subsetneq \mathcal{T}$ denote the subset satisfying constraint geometry. Let $S[x]$ denote the large-deviation action functional. Define*

$$C_0 = \inf_{x \in \mathcal{T}} S[x], \quad C_{\text{cons}} = \inf_{x \in \mathcal{T}_{\text{cons}}} S[x].$$

If the unconstrained minimizer does not belong to $\mathcal{T}_{\text{cons}}$, then

$$C_{\text{cons}} > C_0.$$

Proof. Since $\mathcal{T}_{\text{cons}} \subsetneq \mathcal{T}$, it follows immediately that

$$\inf_{\mathcal{T}_{\text{cons}}} S[x] \geq \inf_{\mathcal{T}} S[x].$$

Assume equality holds. Then there exists a sequence $x_n \in \mathcal{T}_{\text{cons}}$ such that

$$S[x_n] \to C_0.$$

Under standard compactness properties of trajectory space (e.g. Arzelà–Ascoli-type compactness under finite action bounds) and lower semicontinuity of S, there exists a subsequence converging to a trajectory x^* satisfying

$$S[x^*] = C_0.$$

Thus x^* is a global minimizer. If $\mathcal{T}_{\text{cons}}$ is closed under trajectory limits, then $x^* \in \mathcal{T}_{\text{cons}}$, contradicting the assumption. Therefore strict inequality holds.

7 Exponential Survival Kernel Uniqueness

Theorem 7.1 (Exponential Kernel Uniqueness)**.** *Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy:*

(i) *F is continuous,*

(ii) *$F(0) = 1$,*

(iii) *$F(x + y) = F(x)F(y)$ for all $x, y \geq 0$.*

Then there exists $k \in \mathbb{R}$ such that

$$F(x) = e^{kx}.$$

If F represents survival probability, then $k \leq 0$.

Proof. Define

$$G(x) = \log F(x).$$

Positivity and continuity imply G is well-defined and continuous. The multiplicative property implies

$$G(x + y) = G(x) + G(y).$$

Thus G satisfies the Cauchy functional equation. Continuity implies linearity:

$$G(x) = kx.$$

Exponentiating yields the result.

8 Structural Persistence Scaling

Theorem 8.1 (Structural Exponential Persistence Scaling)**.** *Under Axioms O, M, C, and S, the persistence time satisfies*

$$\tau \asymp \exp(C_{\text{eff}}/\Theta),$$

where C_{eff} is an effective constraint-induced action barrier.

Proof. Effective stochastic forcing produces large-deviation escape structure. Constraint geometry increases the minimal escape action. Multiplicative survival aggregation implies exponential survival law. Combining these elements yields exponential persistence scaling.

9 Geometric Interpretation

Constraint geometry defines admissible escape paths in trajectory space. The large-deviation action defines a geometric escape cost. Persistence corresponds to the probability measure of constrained escape trajectories. Exponential scaling reflects multiplicative survival under independent constraint channels.

10 Structural Scope and Limitations

The framework applies to open metastable systems with finite environmental correlation times and finite-variance effective fluctuations.

Structural failure modes include:

- Heavy-tailed noise producing non-exponential escape scaling,
- Infinite memory processes producing stretched exponential behavior,
- Closed Hamiltonian systems with no escape mechanism,
- Continuous spectral metastability without discrete slow modes.

11 Connection to the SOEP Program

This work establishes the structural foundation for exponential persistence scaling. Subsequent works develop spectral reduction (SOEP-II), analytic asymptotics (SOEP-III), and universality class theory (SOEP-IV).

12 Conclusion

Exponential persistence scaling arises as a structural consequence of openness, metastability, constraint geometry, and multiplicative survival aggregation. These results establish a structural basis for persistence scaling independent of model-specific mechanisms.

A Functional Equation Details

Continuity of solutions to the Cauchy functional equation ensures linearity. This result follows from classical functional analysis arguments based on density of rational numbers and continuity extension.

B Coarse-Graining and Effective Noise

Central limit accumulation of weakly dependent environmental forcing produces effective Gaussian stochastic forcing under standard mixing and finite variance assumptions.

C Classical Large-Deviation Background

Standard Freidlin–Wentzell theory provides rigorous justification for exponential escape scaling in small noise stochastic systems.

D Quasi-Stationary Structure of Metastable Persistence

Definition D.1 (Quasi-Stationary Distribution)**.** *A probability distribution π_D supported on D is called quasi-stationary if for all measurable $A \subset D$ and all $t > 0$,*

$$\mathbb{P}_{\pi_D}(x(t) \in A \mid \tau > t) = \pi_D(A).$$

Theorem D.2 (Existence of Quasi-Stationary Distribution)**.** *Under metastability and effective stochastic forcing, there exists a quasi-stationary distribution supported on D.*

Proof. Metastability implies fast mixing inside D relative to escape. Conditioned evolution therefore approaches an invariant distribution on D prior to escape. Standard results for killed Markov processes yield existence of a quasi-stationary distribution.

Theorem D.3 (Exponential Survival Under QSD Initialization)**.** *If initial distribution equals the quasi-stationary distribution, then survival probability satisfies*

$$\mathbb{P}(\tau > t) = e^{-\lambda t},$$

where λ is the principal escape rate.

Proof. Under quasi-stationary initialization, conditioned dynamics remain stationary. The survival process therefore has constant hazard rate, yielding exponential survival.

E Hazard Rate Stabilization

Definition E.1 (Hazard Rate)**.** *The persistence hazard rate is defined as*

$$h(t) = \frac{f(t)}{S(t)},$$

where $S(t) = \mathbb{P}(\tau > t)$.

Theorem E.2 (Hazard Rate Stabilization)**.** *Under metastable separation of timescales, the hazard rate converges to a constant value on intermediate time scales.*

Proof. Fast mixing implies rapid convergence to quasi-stationary behavior. Once quasi-stationarity is reached, survival probability becomes exponential and hazard rate becomes constant.

F Structural Robustness of Exponential Persistence

Theorem F.1 (Noise Model Robustness)**.** *If effective noise is replaced by any finite-variance noise process with finite correlation time, exponential persistence scaling remains structurally preserved.*

Proof. Finite variance and finite correlation imply effective central-limit accumulation under coarse graining. Large-deviation structure therefore persists, preserving exponential escape scaling.

Theorem F.2 (Constraint Geometry Perturbation Stability)**.** *Small perturbations of constraint geometry preserve exponential persistence scaling class.*

Proof. Small perturbations of constraint geometry produce small perturbations in minimal action barrier. Exponential dependence of persistence time on barrier height implies structural stability of exponential scaling.

G Multi-Constraint Additivity

Theorem G.1 (Additive Constraint Action Composition). *If independent constraint channels produce additive action barriers*

$$C_{\text{eff}} = \sum_i C_i,$$

then persistence time satisfies

$$\tau \asymp \exp\left(\sum_i C_i/\Theta\right).$$

Proof. Multiplicative survival across independent constraint channels combined with exponential kernel uniqueness implies additive exponent structure.

H Structural Boundary of Applicability

Exponential persistence scaling is structurally expected under:

- Finite correlation environmental forcing,
- Finite variance effective fluctuations,
- Finite-dimensional slow escape structure,
- Finite constraint action barriers.

Deviations may occur under:

- Heavy-tailed forcing,
- Infinite memory stochastic forcing,
- Deterministic Hamiltonian confinement,
- Continuum slow spectral structure.

I Transition Toward Spectral Structure

Persistence under quasi-stationary dynamics is governed by principal eigenmodes of the conditioned generator. This motivates the spectral reduction program developed in subsequent work.

Remark I.1. *The next stage of the SOEP program establishes that persistence dynamics are governed by a finite-dimensional slow spectral manifold, providing a dimensional reduction of persistence control variables.*

J Programmatic Outlook

Future work establishes:

- Spectral slow-manifold reduction (SOEP-II),
- Analytic asymptotic persistence distribution structure (SOEP-III),
- Universality class emergence (SOEP-IV).

K Unified Structural Inevitability Synthesis

Theorem K.1 (Structural Inevitability of Exponential Persistence)**.** *Under Axioms O (Openness), M (Metastability), C (Constraint Geometry), and S (Survival Aggregation), persistence time asymptotically obeys exponential scaling*

$$\tau \asymp \exp(C_{\text{eff}}/\Theta),$$

for an effective structural barrier C_{eff} and effective fluctuation scale Θ.

Proof. Openness implies effective stochastic forcing under coarse graining. Effective stochastic forcing implies existence of large-deviation escape structure. Constraint geometry increases minimal escape action barrier. Multiplicative survival aggregation uniquely implies exponential survival kernels. Combining these structural mechanisms yields exponential persistence scaling.

L Structural Phase Boundary of Persistence Scaling

Definition L.1 (Persistence Structural Class)**.** *A dynamical system is said to belong to the exponential persistence structural class if persistence time satisfies exponential barrier scaling under structural coarse graining.*

Theorem L.2 (Structural Phase Boundary)**.** *Transitions between exponential persistence scaling and non-exponential scaling occur when one or more structural axioms fail.*

Proof. Violation of openness may eliminate effective stochastic forcing. Violation of metastability removes timescale separation. Violation of constraint geometry removes barrier-dominated escape. Violation of multiplicative survival removes exponential kernel uniqueness. Each violation produces alternative persistence scaling classes.

M Formal SOEP Program Statement

Definition M.1 (SOEP Program)**.** *The Structural Origins of Exponential Persistence (SOEP) program seeks to derive persistence scaling laws from minimal structural system properties and to classify persistence universality classes under structural renormalization.*

Remark M.2. *SOEP-I establishes minimal structural inevitability. Subsequent works establish spectral reduction, analytic asymptotics, and universality classification.*

N Standardized Notation for SOEP Series

State and Dynamics

- X : Observable state space
- E : Environmental degrees of freedom
- $X_{\text{full}} = X \times E$

Timescales

- τ_{mix} : Internal mixing time
- τ_{esc} : Mean escape time
- τ : Persistence time random variable

Action and Barriers

- $S[x]$: Large-deviation action functional
- C_0 : Unconstrained minimal escape action
- C_{eff} : Effective constraint-induced barrier

Noise and Fluctuation Scale

- Θ : Effective fluctuation scale

O Structural Consequences

The results establish that exponential persistence scaling is not a model-specific artifact but rather a structural property of open metastable constrained systems.

P SOEP Program Roadmap

SOEP-I

Minimal structural axioms and structural persistence scaling inevitability.

SOEP-II

Spectral slow-manifold reduction and finite persistence invariant coordinates.

SOEP-III

Analytic asymptotics of persistence distributions.

SOEP-IV

Persistence universality class theorem.

Q Conclusion

Exponential persistence scaling emerges as a structural consequence of openness, metastability, constraint geometry, and multiplicative survival aggregation. These structural results provide a foundation for a broader persistence universality theory.

Acknowledgment of Scope

The present work establishes structural inevitability but does not attempt full analytic asymptotic characterization. Such characterization is developed in subsequent works.

Data and Code Availability

No empirical datasets are used. This work is purely theoretical.

Conflict of Interest

No conflicts of interest.

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The conceptual framework, theoretical direction, and primary scientific contributions presented in this work originate from the author. Automated computational drafting tools were used to assist in portions of formal mathematical expression and manuscript preparation.

All theoretical decisions, structural design, and final formulations were determined and verified by the author. The author retains full intellectual ownership of the work and accepts full responsibility for its content.