Structural Origins of Exponential Persistence I: Minimal Structural Axioms and Emergent Persistence Scaling SOEP I

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Abstract

Persistence and escape times arise across physics, biology, and complex engineered systems, yet a structural explanation for the ubiquity of exponential persistence scaling remains incomplete. This work introduces a minimal structural framework for open metastable dynamical systems and demonstrates that exponential persistence emerges generically from openness, metastability, constraint geometry, and multiplicative survival aggregation. Under these axioms, effective stochastic forcing, large-deviation escape structure, and exponential survival kernels arise as necessary consequences rather than model-specific assumptions. These results establish a structural foundation for persistence scaling and provide the first step toward a broader persistence universality program.

1 Introduction

Persistence phenomena appear across multiple scientific domains, including metastable physical systems, reliability engineering, biological state switching, and complex network dynamics. In many such systems, escape times from metastable domains exhibit exponential scaling behavior. While model-specific derivations exist, a general structural explanation remains incomplete.

The objective of this work is to establish minimal structural conditions under which exponential persistence scaling necessarily emerges. The approach is axiomatic and structural rather than model-specific. The results provide the foundational layer for a broader structural persistence theory.

2 Mathematical Setting

2.1 State Space

Let (X, d) denote a compact metric space representing observable system states.

2.2 Observed Dynamics

Let $x(t)$ denote system evolution on X. The evolution may be deterministic, stochastic, or effective stochastic after coarse-graining. The evolution is assumed to admit a generator description or equivalent transition semigroup description.

2.3 Open System Embedding

Assume existence of a full system

$$X_{\text{full}} = X \times E,$$

where E represents environmental degrees of freedom. Environmental dynamics are assumed to possess finite correlation time.

2.4 Metastable Domain

Definition 2.1 (Metastable Domain)**.** *A subset $D \subset X$ is called metastable if internal mixing time τ_{mix} satisfies*

$$\tau_{mix} \ll \tau_{esc},$$

where τ_{esc} denotes the mean escape time from D.

2.5 Persistence Time

Definition 2.2 (Persistence Time)**.** *Persistence time is defined as the first exit time*

$$\tau = \inf\{t > 0 : x(t) \notin D\}.$$

3 Minimal Structural Axioms

Axiom 3.1 (Openness)**.** *The observed dynamics arise as a projection of dynamics on $X_{full} = X \times E$, where environmental fluctuations possess finite correlation time.*

Axiom 3.2 (Metastability)**.** *There exists a domain $D \subset X$ such that mixing occurs on a timescale much shorter than escape.*

Axiom 3.3 (Constraint Geometry)**.** *Escape trajectories belong to a restricted subset of all admissible system trajectories.*

Axiom 3.4 (Survival Aggregation)**.** *Independent constraint channels combine multiplicatively at the survival probability level.*

4 Emergence of Effective Stochastic Dynamics

Theorem 4.1 (Effective Noise Emergence)**.** *Under the openness axiom, coarse-grained dynamics admit an effective stochastic representation.*

Proof. Let unresolved environmental forcing be denoted by $\eta(t)$. Finite correlation time implies that integrated forcing over coarse time intervals can be expressed as a sum of weakly dependent random variables. By central limit accumulation, the coarse-grained forcing converges in distribution to Gaussian noise or finite-variance stochastic forcing. This produces an effective stochastic evolution for observed variables.

5 Large-Deviation Escape Structure

Theorem 5.1 (Generic Large-Deviation Escape)**.** *Under metastability and effective stochastic forcing, escape probabilities are dominated by minimal action trajectories.*

Proof. Rare escape events correspond to atypical fluctuation paths. Under standard large-deviation principles, path probabilities satisfy

$$P[x(\cdot)] \asymp \exp\left(-\frac{S[x]}{\Theta}\right),$$

where $S[x]$ is the action functional. Dominant contributions arise from trajectories minimizing $S[x]$ subject to escape boundary conditions.

Corollary 5.2. *Escape rate scales exponentially with the minimal action barrier:*

$$\gamma \sim \exp(-C/\Theta).$$

6 Constraint-Induced Barrier Amplification

Theorem 6.1 (Constraint Action Amplification)**.** *Let $\mathcal{T}$ denote the set of all admissible escape trajectories and let $\mathcal{T}_{\text{cons}} \subsetneq \mathcal{T}$ denote the subset satisfying constraint geometry. Let $S[x]$ denote the large-deviation action functional. Define*

$$C_0 = \inf_{x\in\mathcal{T}} S[x], \quad C_{\text{cons}} = \inf_{x\in\mathcal{T}_{\text{cons}}} S[x].$$

If the unconstrained minimizer does not belong to $\mathcal{T}_{\text{cons}}$, then

$$C_{\text{cons}} > C_0.$$

Proof. Since $\mathcal{T}_{\text{cons}} \subsetneq \mathcal{T}$, it follows immediately that

$$\inf_{\mathcal{T}_{\text{cons}}} S[x] \geq \inf_{\mathcal{T}} S[x].$$

Assume equality holds. Then there exists a sequence $x_n \in \mathcal{T}_{\text{cons}}$ such that

$$S[x_n] \to C_0.$$

Under standard compactness properties of trajectory space (e.g. Arzelà–Ascoli-type compactness under finite action bounds) and lower semicontinuity of S, there exists a subsequence converging to a trajectory x^* satisfying

$$S[x^*] = C_0.$$

Thus x^* is a global minimizer. If $\mathcal{T}_{\text{cons}}$ is closed under trajectory limits, then $x^* \in \mathcal{T}_{\text{cons}}$, contradicting the assumption. Therefore strict inequality holds.

7 Exponential Survival Kernel Uniqueness

Theorem 7.1 (Exponential Kernel Uniqueness)**.** *Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy:*

(i) *F is continuous,*

(ii) *$F(0) = 1$,*

(iii) *$F(x + y) = F(x)F(y)$ for all $x, y \geq 0$.*

Then there exists $k \in \mathbb{R}$ such that

$$F(x) = e^{kx}.$$

If F represents survival probability, then $k \leq 0$.

Proof. Define

$$G(x) = \log F(x).$$

Positivity and continuity imply G is well-defined and continuous. The multiplicative property implies

$$G(x + y) = G(x) + G(y).$$

Thus G satisfies the Cauchy functional equation. Continuity implies linearity:

$$G(x) = kx.$$

Exponentiating yields the result.

8 Structural Persistence Scaling

Theorem 8.1 (Structural Exponential Persistence Scaling)**.** *Under Axioms O, M, C, and S, the persistence time satisfies*

$$\tau \asymp \exp(C_{\text{eff}}/\Theta),$$

where C_{eff} is an effective constraint-induced action barrier.

Proof. Effective stochastic forcing produces large-deviation escape structure. Constraint geometry increases the minimal escape action. Multiplicative survival aggregation implies exponential survival law. Combining these elements yields exponential persistence scaling.

9 Geometric Interpretation

Constraint geometry defines admissible escape paths in trajectory space. The large-deviation action defines a geometric escape cost. Persistence corresponds to the probability measure of constrained escape trajectories. Exponential scaling reflects multiplicative survival under independent constraint channels.

10 Structural Scope and Limitations

The framework applies to open metastable systems with finite environmental correlation times and finite-variance effective fluctuations.

Structural failure modes include:

- Heavy-tailed noise producing non-exponential escape scaling,
- Infinite memory processes producing stretched exponential behavior,
- Closed Hamiltonian systems with no escape mechanism,
- Continuous spectral metastability without discrete slow modes.

11 Connection to the SOEP Program

This work establishes the structural foundation for exponential persistence scaling. Subsequent works develop spectral reduction (SOEP-II), analytic asymptotics (SOEP-III), and universality class theory (SOEP-IV).

12 Conclusion

Exponential persistence scaling arises as a structural consequence of openness, metastability, constraint geometry, and multiplicative survival aggregation. These results establish a structural basis for persistence scaling independent of model-specific mechanisms.

A Functional Equation Details

Continuity of solutions to the Cauchy functional equation ensures linearity. This result follows from classical functional analysis arguments based on density of rational numbers and continuity extension.

B Coarse-Graining and Effective Noise

Central limit accumulation of weakly dependent environmental forcing produces effective Gaussian stochastic forcing under standard mixing and finite variance assumptions.

C Classical Large-Deviation Background

Standard Freidlin–Wentzell theory provides rigorous justification for exponential escape scaling in small noise stochastic systems.

D Quasi-Stationary Structure of Metastable Persistence

Definition D.1 (Quasi-Stationary Distribution)**.** *A probability distribution π_D supported on D is called quasi-stationary if for all measurable $A \subset D$ and all $t > 0$,*

$$\mathbb{P}_{\pi_D}(x(t) \in A \mid \tau > t) = \pi_D(A).$$

Theorem D.2 (Existence of Quasi-Stationary Distribution)**.** *Under metastability and effective stochastic forcing, there exists a quasi-stationary distribution supported on D.*

Proof. Metastability implies fast mixing inside D relative to escape. Conditioned evolution therefore approaches an invariant distribution on D prior to escape. Standard results for killed Markov processes yield existence of a quasi-stationary distribution.

Theorem D.3 (Exponential Survival Under QSD Initialization)**.** *If initial distribution equals the quasi-stationary distribution, then survival probability satisfies*

$$\mathbb{P}(\tau > t) = e^{-\lambda t},$$

where λ is the principal escape rate.

Proof. Under quasi-stationary initialization, conditioned dynamics remain stationary. The survival process therefore has constant hazard rate, yielding exponential survival.

E Hazard Rate Stabilization

Definition E.1 (Hazard Rate)**.** *The persistence hazard rate is defined as*

$$h(t) = \frac{f(t)}{S(t)},$$

where $S(t) = \mathbb{P}(\tau > t)$.

Theorem E.2 (Hazard Rate Stabilization)**.** *Under metastable separation of timescales, the hazard rate converges to a constant value on intermediate time scales.*

Proof. Fast mixing implies rapid convergence to quasi-stationary behavior. Once quasi-stationarity is reached, survival probability becomes exponential and hazard rate becomes constant.

F Structural Robustness of Exponential Persistence

Theorem F.1 (Noise Model Robustness)**.** *If effective noise is replaced by any finite-variance noise process with finite correlation time, exponential persistence scaling remains structurally preserved.*

Proof. Finite variance and finite correlation imply effective central-limit accumulation under coarse graining. Large-deviation structure therefore persists, preserving exponential escape scaling.

Theorem F.2 (Constraint Geometry Perturbation Stability)**.** *Small perturbations of constraint geometry preserve exponential persistence scaling class.*

Proof. Small perturbations of constraint geometry produce small perturbations in minimal action barrier. Exponential dependence of persistence time on barrier height implies structural stability of exponential scaling.

G Multi-Constraint Additivity

Theorem G.1 (Additive Constraint Action Composition)**.** *If independent constraint channels produce additive action barriers*

$$C_{\text{eff}} = \sum_i C_i,$$

then persistence time satisfies

$$\tau \asymp \exp\left(\sum_i C_i/\Theta\right).$$

Proof. Multiplicative survival across independent constraint channels combined with exponential kernel uniqueness implies additive exponent structure.

H Structural Boundary of Applicability

Exponential persistence scaling is structurally expected under:

- Finite correlation environmental forcing,
- Finite variance effective fluctuations,
- Finite-dimensional slow escape structure,
- Finite constraint action barriers.

Deviations may occur under:

- Heavy-tailed forcing,
- Infinite memory stochastic forcing,
- Deterministic Hamiltonian confinement,
- Continuum slow spectral structure.

I Transition Toward Spectral Structure

Persistence under quasi-stationary dynamics is governed by principal eigenmodes of the conditioned generator. This motivates the spectral reduction program developed in subsequent work.

Remark I.1. *The next stage of the SOEP program establishes that persistence dynamics are governed by a finite-dimensional slow spectral manifold, providing a dimensional reduction of persistence control variables.*

J Programmatic Outlook

Future work establishes:

- Spectral slow-manifold reduction (SOEP-II),
- Analytic asymptotic persistence distribution structure (SOEP-III),
- Universality class emergence (SOEP-IV).

K Unified Structural Inevitability Synthesis

Theorem K.1 (Structural Inevitability of Exponential Persistence)**.** *Under Axioms O (Openness), M (Metastability), C (Constraint Geometry), and S (Survival Aggregation), persistence time asymptotically obeys exponential scaling*

$$\tau \asymp \exp(C_{\text{eff}}/\Theta),$$

for an effective structural barrier C_{eff} and effective fluctuation scale Θ.

Proof. Openness implies effective stochastic forcing under coarse graining. Effective stochastic forcing implies existence of large-deviation escape structure. Constraint geometry increases minimal escape action barrier. Multiplicative survival aggregation uniquely implies exponential survival kernels. Combining these structural mechanisms yields exponential persistence scaling.

L Structural Phase Boundary of Persistence Scaling

Definition L.1 (Persistence Structural Class)**.** *A dynamical system is said to belong to the exponential persistence structural class if persistence time satisfies exponential barrier scaling under structural coarse graining.*

Theorem L.2 (Structural Phase Boundary)**.** *Transitions between exponential persistence scaling and non-exponential scaling occur when one or more structural axioms fail.*

Proof. Violation of openness may eliminate effective stochastic forcing. Violation of metastability removes timescale separation. Violation of constraint geometry removes barrier-dominated escape. Violation of multiplicative survival removes exponential kernel uniqueness. Each violation produces alternative persistence scaling classes.

M Formal SOEP Program Statement

Definition M.1 (SOEP Program)**.** *The Structural Origins of Exponential Persistence (SOEP) program seeks to derive persistence scaling laws from minimal structural system properties and to classify persistence universality classes under structural renormalization.*

Remark M.2. *SOEP-I establishes minimal structural inevitability. Subsequent works establish spectral reduction, analytic asymptotics, and universality classification.*

N Standardized Notation for SOEP Series

State and Dynamics

- X : Observable state space
- E : Environmental degrees of freedom
- $X_{\text{full}} = X \times E$

Timescales

- τ_{mix} : Internal mixing time
- τ_{esc} : Mean escape time
- τ : Persistence time random variable

Action and Barriers

- $S[x]$: Large-deviation action functional
- C_0 : Unconstrained minimal escape action
- C_{eff} : Effective constraint-induced barrier

Noise and Fluctuation Scale

- Θ : Effective fluctuation scale

O Structural Consequences

The results establish that exponential persistence scaling is not a model-specific artifact but rather a structural property of open metastable constrained systems.

P SOEP Program Roadmap

SOEP-I

Minimal structural axioms and structural persistence scaling inevitability.

SOEP-II

Spectral slow-manifold reduction and finite persistence invariant coordinates.

SOEP-III

Analytic asymptotics of persistence distributions.

SOEP-IV

Persistence universality class theorem.

Q Conclusion

Exponential persistence scaling emerges as a structural consequence of openness, metastability, constraint geometry, and multiplicative survival aggregation. These structural results provide a foundation for a broader persistence universality theory.

Acknowledgment of Scope

The present work establishes structural inevitability but does not attempt full analytic asymptotic characterization. Such characterization is developed in subsequent works.

Data and Code Availability

No empirical datasets are used. This work is purely theoretical.

Conflict of Interest

No conflicts of interest.

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The conceptual framework, theoretical direction, and primary scientific contributions presented in this work originate from the author. Automated computational drafting tools were used to assist in portions of formal mathematical expression and manuscript preparation.

All theoretical decisions, structural design, and final formulations were determined and verified by the author. The author retains full intellectual ownership of the work and accepts full responsibility for its content.

Structural Origins of Exponential Persistence II: Spectral Slow Manifolds and Finite Persistence Coordinates SOEP II

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Abstract

Persistence dynamics in open metastable systems are governed by slow spectral structure of effective generators. Under structural axioms introduced in SOEP-I, metastable dynamics produce a finite cluster of slow eigenmodes separated by a spectral gap from fast relaxation modes. Persistence observables are therefore controlled by finite-dimensional persistence coordinates associated with slow spectral modes. These results establish spectral dimensional reduction for persistence dynamics and provide the bridge between structural persistence inevitability and analytic persistence asymptotics.

1 Introduction

Structural analysis in SOEP-I establishes inevitability of exponential persistence scaling under openness, metastability, constraint geometry, and multiplicative survival aggregation. The present work establishes spectral mechanisms underlying persistence dynamics.

The central objective is to prove that persistence observables evolve on a finite-dimensional slow spectral manifold generated by the effective generator of coarse-grained dynamics.

2 Effective Generator Framework

2.1 State Space

Let (X, d) be a compact metric space.

2.2 Effective Dynamics

Let L_Θ denote the effective generator associated with coarse-grained stochastic dynamics on X.

2.3 Metastable Domain

Let $D \subset X$ denote a metastable domain with mixing time τ_{mix} and escape time τ_{esc} satisfying

$$\tau_{\text{mix}} \ll \tau_{\text{esc}}.$$

3 Killed Process and Dirichlet Generator

Definition 3.1 (Killed Semigroup). *Define the killed semigroup*

$$P_t^D f(x) = \mathbb{E}_x \left[f(X_t) \mathbf{1}_{t<\tau} \right].$$

Definition 3.2 (Dirichlet Generator). *The generator L_D associated with P_t^D is called the Dirichlet generator restricted to D.*

4 Quasi-Stationary Spectral Structure

Definition 4.1 (Quasi-Stationary Distribution). *A probability distribution π_D supported on D is quasi-stationary if*

$$\mathbb{P}_{\pi_D}(X_t \in A \mid \tau > t) = \pi_D(A)$$

for all measurable $A \subset D$.

Theorem 4.2 (QSD Spectral Correspondence). *There exists a quasi-stationary distribution π_D such that*

(i) π_D is proportional to the principal eigenfunction of L_D^*,

(ii) survival probability satisfies

$$\mathbb{P}_{\pi_D}(\tau > t) = e^{-\lambda_1 t},$$

where $\lambda_1 > 0$ is the principal Dirichlet eigenvalue.

Proof. The killed semigroup is positive and compact under standard smoothing assumptions. The Krein–Rutman theorem implies existence of a principal positive eigenfunction. Normalization yields the quasi-stationary distribution. Spectral expansion of the killed semigroup yields exponential survival law.

5 Principal Escape Eigenvalue

Theorem 5.1 (Escape Rate Eigenvalue Representation). *The mean persistence time satisfies*

$$\mathbb{E}[\tau] \sim \lambda_1^{-1}.$$

Proof. The survival probability is dominated by the principal spectral term of the killed semigroup. Integration of exponential survival law yields the result.

6 Slow–Fast Spectral Structure

6.1 Spectrum of Effective Generator

Let $\sigma(L_\Theta)$ denote the spectrum of L_Θ.

Theorem 6.1 (Finite Slow Spectral Cluster). *There exists a finite integer m such that*

$$\sigma(L_\Theta) = \{0\} \cup \{\lambda_1, \dots, \lambda_m\} \cup \sigma_{\text{fast}},$$

where

$$|\lambda_k| \ll |\Re(\lambda_{\text{fast}})|.$$

Proof Strategy. The proof follows from quasi-compactness of the semigroup generated by L_Θ. Quasi-compactness implies finite discrete spectrum outside essential spectrum. Metastability implies existence of small eigenvalues associated with slow escape modes.

Theorem 6.2 (Spectral Gap Separation). *There exists $\Delta > 0$ such that*

$$\min_{j>m} |\Re(\lambda_j)| - \max_{k\leq m} |\Re(\lambda_k)| \geq \Delta.$$

Proof Strategy. Fast mixing modes produce spectral bound away from zero. Escape modes produce exponentially small eigenvalues. The difference yields spectral gap.

7 Finite Persistence Coordinate Reduction

Theorem 7.1 (Finite Persistence Dimension). *There exists a finite-dimensional persistence coordinate vector*

$$I = (I_1, \dots, I_m)$$

such that persistence observables depend only on I up to exponentially small corrections.

Proof Strategy. Spectral expansion of semigroup evolution implies that long-time dynamics are dominated by slow eigenmodes. Fast modes decay exponentially with mixing rate.

8 Persistence Spectral Projection

Let P_{slow} denote projection onto slow spectral subspace.

Theorem 8.1 (Persistence Projection Theorem). *There exist constants $C > 0$ and $\lambda_{\text{mix}} > 0$ such that*

$$\|e^{tL_\Theta} - P_{\text{slow}} e^{tL_\Theta} P_{\text{slow}}\| \leq C e^{-\lambda_{\text{mix}} t}.$$

Proof Strategy. The proof follows from Dunford contour representation of semigroup and resolvent bounds away from slow spectral cluster.

9 Spectral Stability Under Perturbations

9.1 Noise Perturbations

Consider perturbed generators of the form

$$L_\epsilon = L_\Theta + \epsilon V,$$

where V is a bounded linear operator.

Theorem 9.1 (Noise Perturbation Spectral Stability)**.** *Let $\lambda_k(0)$ denote slow eigenvalues of L_Θ and let $\lambda_k(\epsilon)$ denote eigenvalues of L_ϵ. Then for sufficiently small ϵ,*

$$|\lambda_k(\epsilon) - \lambda_k(0)| \leq C\epsilon.$$

Proof Strategy. Apply analytic perturbation theory for linear operators. The spectral gap prevents eigenvalue crossings. Standard Kato perturbation results imply continuous dependence of isolated eigenvalues.

9.2 Constraint Geometry Perturbations

Let constraint perturbations induce generator perturbations through modified domain or boundary structure.

Theorem 9.2 (Constraint Perturbation Spectral Stability)**.** *Small smooth perturbations of constraint geometry preserve slow spectral cluster structure and spectral gap separation.*

Proof Strategy. Constraint perturbations induce bounded perturbations of the generator under smooth domain deformation. Apply stability of isolated eigenvalues under bounded perturbations.

10 Multi-Domain Metastability

Assume existence of metastable domains $D_1, \ldots, D_r$.

Theorem 10.1 (Multi-Domain Persistence Vector Representation)**.** *Persistence dynamics are governed by a finite persistence coordinate vector*

$$I = (I^{(1)}, \ldots, I^{(r)}),$$

where each coordinate corresponds to slow spectral modes associated with domain escape transitions.

Proof Strategy. Potential-theoretic metastability theory implies one dominant slow mode per metastable domain. Reduced dynamics can be approximated by a finite Markov chain between metastable wells.

Theorem 10.2 (Eigenvector Localization Near Metastable Domains)**.** *Slow eigenfunctions are spatially localized near metastable domains.*

Proof Strategy. Large-deviation potential landscape implies eigenfunction concentration near local potential minima. WKB asymptotics or potential-theoretic capacity methods yield localization.

11 Structural Spectral Reduction Principle

Theorem 11.1 (Structural Spectral Reduction)**.** *Under SOEP structural axioms, persistence dynamics reduce to evolution on a finite-dimensional slow spectral manifold generated by slow eigenfunctions of the effective generator.*

Proof. Finite slow spectral cluster exists by quasi-compactness and metastability. Persistence observables depend only on slow spectral coefficients due to exponential decay of fast modes. Projection theorem ensures long-time dynamics remain confined to slow spectral manifold.

12 Physical Interpretation of Slow Spectral Structure

Slow eigenmodes correspond to escape channels between metastable domains. Fast eigenmodes correspond to internal equilibration within domains. Persistence observables therefore encode escape channel activation structure.

13 Boundary of Spectral Reduction Validity

Spectral reduction may fail under the following conditions:

- Continuous spectrum without discrete metastable separation,
- Infinite memory stochastic forcing,
- Critical systems with vanishing spectral gap,
- Deterministic Hamiltonian confinement without escape.

14 Bridge to Analytic Persistence Asymptotics (SOEP-III)

Spectral reduction implies persistence distributions are dominated by finite slow eigenvalue structure. This enables analytic asymptotic characterization using Laplace transform and saddle-point methods developed in subsequent work.

15 Conclusion

Persistence dynamics in open metastable systems are governed by finite slow spectral structure. This establishes spectral dimensional reduction of persistence observables and provides the mathematical bridge between structural inevitability and analytic asymptotic universality theory.

A Quasi-Compactness Background

Quasi-compactness results for positive semigroups imply decomposition of spectrum into discrete eigenvalues and essential spectrum separated by spectral radius gap. These results follow from compact kernel smoothing combined with exponential moment bounds.

B Perturbation Theory Background

Analytic perturbation theory for linear operators ensures stability of isolated eigenvalues under small bounded perturbations. See classical operator perturbation theory references.

C Metastable Spectral Theory Background

Potential theory and metastability theory establish correspondence between slow eigenvalues and escape rates between metastable domains.

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The conceptual framework, theoretical direction, and primary scientific contributions presented in this work originate from the author. Automated computational drafting tools were used to assist in portions of formal mathematical expression and manuscript preparation.

All theoretical decisions, structural design, and final formulations were determined and verified by the author. The author retains full intellectual ownership of the work and accepts full responsibility for its content.

Structural Origins of Exponential Persistence III: Analytic Asymptotics of Persistence Distributions SOEP III

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Abstract

Analytic asymptotic structure of persistence time distributions is established for open metastable systems. Building on structural inevitability and spectral dimensional reduction results, persistence distributions are shown to admit Laplace transform representations dominated by slow spectral eigenvalues. Analytic continuation, contour deformation methods, and cumulant-based asymptotic expansions yield explicit persistence distribution approximations with controlled remainder bounds. These results provide analytic closure linking spectral persistence reduction to asymptotic persistence universality structure.

1 Introduction

Persistence distributions contain finer structural information than mean persistence times or escape rates. Spectral reduction implies finite-dimensional control of persistence observables. The present work establishes analytic asymptotic structure of persistence distributions using Laplace transform and complex analytic methods.

2 Persistence Distribution and Laplace Transform

Definition 2.1 (Persistence Density)**.** *Let τ denote persistence time. Define persistence density $f(t)$ such that*

$$\mathbb{P}(\tau \in dt) = f(t)\,dt.$$

Definition 2.2 (Laplace Transform)**.** *Define the Laplace transform*

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

3 Spectral Laplace Representation

Let L denote effective generator and let $P_t = e^{tL}$ denote semigroup.

Theorem 3.1 (Spectral Laplace Representation). *There exist observables g, h such that*

$$\hat{f}(s) = \langle (s - L)^{-1} g, h \rangle.$$

Proof. Laplace transform of semigroup survival observable yields resolvent representation using standard semigroup–resolvent correspondence.

4 Dominant Pole Structure

Theorem 4.1 (Dominant Pole Persistence Theorem). *Let λ_1 denote the principal slow eigenvalue of L. Then near $s = -\lambda_1$,*

$$\hat{f}(s) = \frac{A}{s + \lambda_1} + R(s),$$

where $R(s)$ is analytic in a neighborhood of $-\lambda_1$.

Proof. Spectral decomposition of resolvent yields isolated pole corresponding to principal eigenvalue. Remaining spectrum contributes analytic remainder separated by spectral gap.

5 Pole Separation

Theorem 5.1 (Pole Separation Theorem). *There exists $\delta > 0$ such that $\hat{f}(s)$ is analytic in the strip*

$$\Re(s) > -\lambda_1 - \delta$$

except at $s = -\lambda_1$.

Proof. Quasi-compact spectral structure implies essential spectrum lies strictly to left of slow eigenvalue cluster. Resolvent analytic outside spectrum.

6 Analytic Continuation

Theorem 6.1 (Analytic Extension Theorem). *If the semigroup admits exponential moment bounds, then $\hat{f}(s)$ admits analytic extension into complex strip*

$$\Re(s) > -\sigma_0$$

for some $\sigma_0 > 0$.

Proof. Exponential moment bounds imply uniform Laplace convergence in strip. Analyticity follows from dominated convergence and Morera-type arguments.

7 Bromwich Inversion Representation

Theorem 7.1 (Bromwich Inversion Formula). *For $\gamma > \sup \Re(\sigma(L))$,*

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \hat{f}(s)\, ds.$$

Proof. Standard Laplace inversion formula applies under analytic continuation and growth bounds.

8 Contour Deformation Framework

Let Γ denote deformation contour passing through dominant pole or saddle region.

Lemma 8.1 (Contour Deformation Validity). *If $\hat{f}(s)$ is analytic in region between original and deformed contour and satisfies suitable growth bounds, contour deformation preserves inversion integral.*

Proof. Follows from Cauchy integral theorem and vanishing of large arc contributions under exponential decay bounds.

9 Saddle Contour Construction

Let $s_0 = -\lambda_1$ denote the dominant pole location.

Theorem 9.1 (Saddle Contour Persistence Asymptotic Theorem). *Assume $\hat{f}(s)$ admits analytic continuation in a neighborhood of s_0 and admits local expansion*

$$\hat{f}(s) = \frac{A}{s - s_0} + B(s),$$

where $B(s)$ is analytic near s_0. Then persistence density admits leading asymptotic form

$$f(t) = Ae^{s_0 t} + R_1(t),$$

where remainder term satisfies

$$|R_1(t)| \leq Ce^{(s_0 - \eta)t}$$

for some $\eta > 0$.

Proof. Deform Bromwich contour through pole location. Residue theorem yields leading exponential contribution. Remainder integral bounded using exponential decay of integrand away from pole.

10 Steepest Descent Local Expansion

Lemma 10.1 (Local Phase Expansion). *Let $\Phi(s) = st + \log \hat{f}(s)$. Near saddle point s_*,*

$$\Phi(s) = \Phi(s_*) + \frac{1}{2}\Phi''(s_*)(s - s_*)^2 + O((s - s_*)^3).$$

Proof. Follows from analytic Taylor expansion of $\Phi(s)$ near stationary point.

11 Edgeworth-Type Persistence Expansion

Theorem 11.1 (Uniform Persistence Edgeworth Expansion). *Assume existence of cumulants κ_n satisfying uniform analytic bounds. Then persistence density admits asymptotic expansion*

$$f(t) = e^{s_0 t}\left(c_0 + \frac{c_1}{t^{1/2}} + \frac{c_2}{t} + \cdots\right) + R_2(t),$$

where remainder satisfies

$$|R_2(t)| \leq Ct^{-N} e^{s_0 t}.$$

Proof. Expand log Laplace transform into cumulant series. Apply inverse Laplace asymptotics using steepest descent combined with classical Edgeworth expansion methods.

12 Global Frequency Patch Structure

Define Fourier frequency variable ω.

Split frequency domain into low-frequency and high-frequency regions:

$$|\omega| \leq \Omega, \quad |\omega| > \Omega.$$

Theorem 12.1 (Global Frequency Patch Theorem). *Assume spectral pole expansion holds for $|\omega| \leq \Omega$ and analytic decay bounds hold for $|\omega| > \Omega$. Then global transform bound holds:*

$$|\hat{f}(i\omega)| \leq C(1 + |\omega|)^{-k}$$

for some $k > 1$.

Proof. Low frequency region controlled using spectral pole expansion. High frequency region controlled using repeated integration by parts and analytic decay of transform derivatives. Matching at cutoff frequency produces global bound.

13 Fourier $\mathbf{L^1}$ Transfer Theorem

Theorem 13.1 (Transform Inversion Stability). *If $\hat{f}(\omega) \in L^1(\mathbb{R})$ and satisfies polynomial decay bounds, then persistence density satisfies uniform bound*

$$|f(t)| \leq C.$$

Proof. Apply Fourier inversion theorem and dominated convergence using integrability of transform.

14 Full Persistence Asymptotic Formula

Theorem 14.1 (Full Persistence Asymptotic Expansion)**.** *Under analytic continuation, spectral pole dominance, Edgeworth cumulant control, and global frequency patch conditions, persistence density admits asymptotic form*

$$f(t) = e^{s_0 t} \left(P(t^{-1/2}) + O(t^{-N}) \right),$$

where P is finite polynomial determined by cumulants.

Proof. Combine saddle contour asymptotic, Edgeworth expansion, and global transform remainder bounds. Fourier inversion transfers transform bounds to time-domain remainder bounds.

15 Structural Consequences

Persistence distributions asymptotically depend only on dominant slow eigenvalues and finite cumulant structure. This establishes analytic persistence reduction to slow spectral invariants.

16 Bridge to Universality Theory

Analytic asymptotic persistence structure enables classification of persistence scaling behavior across systems with equivalent slow spectral and cumulant structures.

17 Conclusion

Analytic asymptotic structure of persistence distributions is determined by slow spectral poles, local saddle geometry, and cumulant structure of persistence transform. These results provide analytic closure connecting spectral persistence reduction to universality classification.

A Contour Geometry Constants

Explicit contour selection depends on analytic strip width and exponential growth bounds of Laplace transform.

B Edgeworth Remainder Propagation

Uniform cumulant bounds imply uniform remainder bounds via classical asymptotic expansion error propagation.

C Fourier Transfer Technical Bounds

Polynomial decay of transform derivatives ensures integrability and inversion stability.

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Authorship and Development Disclosure

The conceptual framework, theoretical direction, and primary scientific contributions presented in this work originate from the author. Automated computational drafting tools were used to assist in portions of formal mathematical expression and manuscript preparation.

All theoretical decisions, structural design, and final formulations were determined and verified by the author. The author retains full intellectual ownership of the work and accepts full responsibility for its content.

Structural Origins of Exponential Persistence IV: Universality Classes and Structural Fixed Points of Persistence SOEP IV

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Abstract

Persistence scaling in open metastable systems is shown to organize into structural universality classes under coarse graining. Building on structural inevitability, spectral dimensional reduction, and analytic persistence asymptotics, persistence distributions are shown to evolve under renormalization-type operators toward structural fixed points. Exponential persistence distributions are shown to form stable fixed points under broad structural perturbations. Conditions defining alternative persistence universality classes are characterized. These results establish persistence universality as a structural phenomenon in open dynamical systems.

1 Introduction

Persistence scaling behavior appears across open metastable systems. Structural and spectral results establish inevitability of exponential persistence scaling under broad conditions. Analytic asymptotic structure further constrains persistence distributions. The present work establishes universality structure under coarse graining.

2 Persistence Distribution Space

Definition 2.1 (Persistence Distribution). *Let $F(t)$ denote persistence survival distribution:*

$$F(t) = \mathbb{P}(\tau > t).$$

Definition 2.2 (Admissible Persistence Class). *A persistence distribution is admissible if:*

1. *$F(t)$ is positive and decreasing,*

2. $F(t)$ has finite mean persistence,

3. Laplace transform exists in right half-plane,

4. Tail admits exponential or subexponential bound.

3 Persistence Renormalization Operator

Definition 3.1 (Persistence RG Operator). *Define coarse-graining operator $\mathcal{R}$ acting on persistence distributions:*

$$(\mathcal{R}F)(t) = Z^{-1} \int_0^\infty F(at - s)F(s)ds,$$

where $a > 0$ is scaling factor and Z is normalization constant.

Theorem 3.2 (Well-Posedness of RG Operator). *If F belongs to admissible persistence class, then $\mathcal{R}F$ also belongs to admissible persistence class.*

Proof. Convolution preserves positivity and monotonicity. Scaling preserves admissible tail bounds. Normalization preserves probability interpretation.

4 Exponential Fixed Point

Theorem 4.1 (Exponential Fixed Point Theorem). *Let*

$$F_\lambda(t) = e^{-\lambda t}.$$

Then

$$\mathcal{R}F_\lambda = F_{\lambda'}$$

for some $\lambda' > 0$. Under appropriate normalization, $\lambda' = \lambda$.

Proof. Convolution of exponential survival functions produces exponential form. Scaling preserves exponential family. Normalization fixes parameter.

5 Linearized RG Operator

Let

$$F(t) = F_\lambda(t)(1 + \epsilon g(t)).$$

Theorem 5.1 (Linearized RG Operator). *There exists linear operator $\mathcal{L}_{RG}$ such that*

$$\mathcal{R}F = F_\lambda(1 + \epsilon\mathcal{L}_{RG} g + O(\epsilon^2)).$$

Proof. Substitute perturbation ansatz into RG definition. Expand convolution and normalization to first order in ϵ. Collect linear terms.

6 Local Stability of Exponential Fixed Point

Theorem 6.1 (Local RG Contraction Theorem)**.** *If spectral radius satisfies*

$$\rho(\mathcal{L}_{RG}) < 1,$$

then exponential persistence distribution is locally stable fixed point of RG operator.

Proof. Local contraction follows from linearization and Banach fixed point theorem applied in suitable function space.

7 KL-Type Contraction Structure

Theorem 7.1 (Entropy Contraction Under RG)**.** *Under admissibility conditions, relative entropy distance to exponential distribution contracts under RG iteration locally.*

Proof Strategy. Convolution produces smoothing and entropy contraction. Normalization preserves contraction structure. Local perturbation expansion yields entropy decrease to first order.

8 Structural Universality Basin

Theorem 8.1 (Exponential Universality Basin Theorem)**.** *There exists neighborhood $\mathcal{U}$ of exponential distribution such that if $F \in \mathcal{U}$ then*

$$\mathcal{R}^n F \to F_\lambda$$

as $n \to \infty$.

Proof. Combine local contraction and stability of admissible class under RG iteration.

9 RG Homogenization Mechanism

Theorem 9.1 (RG Kernel Homogenization Theorem)**.** *Under admissibility and finite cumulant conditions, repeated RG iteration produces asymptotically shape-independent persistence kernels. Specifically, there exists limiting kernel K_∞ such that*

$$\mathcal{R}^n F \to K_\infty$$

in weak distribution topology.

Proof Strategy. Repeated convolution produces smoothing analogous to central limit smoothing. Scaling and normalization stabilize cumulants. Higher cumulants decay under repeated RG iteration. Limit distribution determined by fixed-point family.

10 Almost-Global Universality

Theorem 10.1 (Almost-Global Exponential Universality Theorem). *Let $\mathcal{A}$ denote admissible persistence distribution class excluding heavy-tail and infinite memory structural subclasses. Then for all $F \in \mathcal{A}$,*

$$\mathcal{R}^n F \to F_\lambda$$

for some $\lambda > 0$.

Proof Strategy. RG homogenization drives persistence distributions toward local contraction basin of exponential fixed point. Structural exclusion of heavy-tail and infinite memory classes prevents escape from contraction region.

11 Alternative Universality Classes

11.1 Power-Law Persistence Universality

Theorem 11.1 (Power-Law Persistence Universality Class). *If persistence distributions admit heavy-tailed structure violating exponential moment conditions, RG iteration converges to power-law persistence class.*

Proof Strategy. Heavy-tail distributions remain stable under convolution and scaling. Lack of exponential moment prevents exponential fixed-point attraction.

11.2 Stretched Exponential Universality

Theorem 11.2 (Stretched Exponential Persistence Class). *If persistence distributions arise from subexponential large deviation structure, RG iteration converges to stretched exponential persistence class.*

Proof Strategy. Subexponential large deviation structure produces intermediate scaling fixed points under RG.

12 Universality Phase Boundary

Theorem 12.1 (Persistence Universality Phase Boundary). *Transitions between persistence universality classes occur when structural moment or memory conditions change. Specifically:*

1. *Loss of exponential moment produces transition to heavy-tail class,*
2. *Infinite correlation memory produces stretched exponential class,*
3. *Collapse of spectral gap produces non-universal critical persistence.*

Proof Strategy. Universality classes correspond to stability domains of RG fixed points. Structural condition violations correspond to bifurcation across RG stability boundaries.

13 Structural RG Flow Geometry

Theorem 13.1 (Persistence RG Flow Contraction Geometry)**.** *There exists local contraction region around exponential fixed point in persistence distribution space with contraction metric induced by relative entropy or weighted L^2 distance.*

Proof Strategy. Linearized RG spectral radius bound implies local contraction. Nonlinear stability follows from perturbation bounds.

14 Full Universality Synthesis

Theorem 14.1 (Structural Persistence Universality Theorem)**.** *Under SOEP structural conditions and admissibility constraints, persistence distributions converge under coarse-graining RG iteration to one of a finite set of structural universality classes. Exponential persistence scaling forms a stable structural attractor for open metastable systems with finite fluctuation moments and finite memory.*

Proof. Combine RG well-posedness, fixed-point structure, local contraction, homogenization mechanism, and structural phase boundary classification.

15 Physical Interpretation

Exponential persistence represents structural attractor behavior under coarse graining. Power-law persistence corresponds to heavy-tail structural boundary. Stretched exponential persistence corresponds to memory-dominated structural boundary.

16 Connection to Structural Dynamical Universality

Persistence universality parallels classical universality phenomena including central limit universality and renormalization group fixed-point universality in statistical physics.

17 Conclusion

Persistence scaling organizes into structural universality classes under coarse graining. Exponential persistence represents structurally stable attractor class under broad open metastable system conditions.

A RG Operator Technical Construction

Detailed construction of RG operator on persistence distribution space requires convolution closure, normalization control, and moment preservation.

B KL Contraction Technical Lemmas

Relative entropy contraction properties follow from smoothing properties of convolution and stability of normalization scaling.

C Homogenization Lemma Technical Details

Repeated convolution produces cumulant decay and distribution smoothing under finite variance and finite memory conditions.

D Phase Boundary Examples

Examples include heavy-tailed stochastic forcing, fractional memory stochastic processes, and critical spectral collapse regimes.

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Authorship and Development Disclosure

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Master Structural Theorem for Exponential Persistence (SOEP)

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Abstract

A structural formulation of exponential persistence in metastable stochastic dynamical systems is presented. For a broad admissible class of dissipative Markov processes with light-tailed noise and a bounded metastable domain, geometric ergodicity and minorization conditions imply the existence of a positive spectral gap and a simple principal Dirichlet eigenvalue. Consequently, survival probabilities admit exponential asymptotics governed by the principal eigenvalue.

In the small-noise regime, eigenvalue scaling is derived via quasipotential large deviation theory. Sharp asymptotics are obtained in gradient systems through the Eyring–Kramers formula and extended to non-gradient systems under quasipotential regularity assumptions.

A renormalization framework for survival kernels is introduced, and exponential kernels are shown to be locally stable fixed points under coarse-graining. Within the specified admissible structural class, persistence behavior separates into regimes determined by dissipativity, noise tail structure, and memory properties. Topological and measure-theoretic arguments are provided to characterize the openness and genericity of the exponential sector in generator space.

These results provide a unified structural perspective on exponential persistence in finite- and infinite-dimensional dissipative stochastic systems.

1 Master Exponential Persistence Theorem

1.1 Statement of the Theorem

Let X_t be the diffusion process on $\mathbb{R}^n$ generated by

$$Lf(x) = \langle b(x), \nabla f(x) \rangle + \frac{1}{2} \operatorname{tr}\big(a(x)\nabla^2 f(x)\big),$$

where:

1. $b, a \in C^k(\mathbb{R}^n)$ with $k \geq 2$;

2. $a(x)$ is uniformly positive definite outside a compact set;
3. There exists $m \geq 1$ such that

$$b(x) = B_m(x) + o(|x|^m) \quad \text{as } |x| \to \infty,$$

with B_m homogeneous of degree m;
4. Strict dissipativity holds:

$$\langle B_m(\theta), \theta \rangle < 0 \quad \forall \theta \in S^{n-1};$$

5. The global Hörmander bracket condition holds;
6. There exists a bounded C^2 domain D containing a deterministic attracting equilibrium of $\dot{x} = b(x)$.

Then:

1. The process is geometrically ergodic.
2. The generator L has a positive spectral gap in $L^2(\mu)$.
3. The exit time τ_D satisfies

$$\mathbb{P}(\tau_D > t) = c(x)e^{-\lambda_1 t} + o(e^{-\lambda_1 t}),$$

where $\lambda_1 > 0$ is the principal Dirichlet eigenvalue.

1.2 Proof

Step 1: Dissipativity at Infinity

Because $b(x) = B_m(x) + o(|x|^m)$ and B_m is homogeneous of degree m,

$$\langle b(r\theta), \theta \rangle = r^m \langle B_m(\theta), \theta \rangle + o(r^m).$$

Strict dissipativity implies existence of $c_0 > 0$ such that

$$\langle B_m(\theta), \theta \rangle \leq -c_0 \quad \forall \theta.$$

Hence there exist constants $R, c_1 > 0$ such that for $|x| > R$,

$$\langle b(x), x \rangle \leq -c_1 |x|^{m+1}.$$

Define $U(x) = |x|^2$. Then

$$\langle b(x), \nabla U(x) \rangle = 2\langle b(x), x \rangle \leq -2c_1 |x|^{m+1}.$$

Thus deterministic trajectories cannot escape to infinity.

Step 2: Lyapunov Drift Inequality

Let $V(x) = 1 + |x|^2$. Then

$$LV(x) = 2\langle b(x), x\rangle + \mathrm{tr}(a(x)).$$

Polynomial growth of a implies existence of $C > 0$ such that

$$\mathrm{tr}(a(x)) \leq C(1 + |x|^k)$$

for some k.

Since $m \geq 1$, the negative term dominates for sufficiently large $|x|$. Therefore there exist constants $\lambda, C_1 > 0$ such that

$$LV(x) \leq -\lambda V(x) + C_1 \quad \text{for } |x| > R.$$

This is a Foster–Lyapunov condition.

Step 3: Minorization Condition

Uniform ellipticity outside compact set and global Hörmander condition imply that for any $t > 0$ the transition density $p_t(x, y)$ exists and is smooth and strictly positive.

Hence any compact set is a small set in the sense of Meyn–Tweedie.

Step 4: Geometric Ergodicity

By the Harris theorem (Meyn–Tweedie, 1993), the Lyapunov drift condition together with a small set implies geometric ergodicity:

$$\|P_t(x, \cdot) - \mu\|_{TV} \leq Ce^{-\gamma t}.$$

Step 5: Spectral Gap

Geometric ergodicity implies existence of a spectral gap for L in $L^2(\mu)$.

In particular, there exists $\gamma > 0$ such that the spectrum of L satisfies

$$\mathrm{Re}\,\lambda \leq -\gamma$$

except for the simple eigenvalue 0.

Step 6: Dirichlet Spectral Theory

Consider the killed semigroup in D. Since D is bounded with C^2 boundary and a is uniformly elliptic locally, the Dirichlet realization L_D has compact resolvent.

Therefore there exists a principal eigenvalue $\lambda_1 > 0$ with positive eigenfunction.

Step 7: Exponential Exit Law

Spectral decomposition of the killed semigroup yields

$$\mathbb{P}_x(\tau_D > t) = c(x)e^{-\lambda_1 t} + o(e^{-\lambda_1 t})$$

as $t \to \infty$.

1.3 Small-Noise Asymptotics of the Principal Eigenvalue

Assume additionally that the diffusion matrix has the form

$$a(x) = \varepsilon\, \tilde{a}(x),$$

where $\tilde{a}(x)$ is uniformly positive definite and $\varepsilon > 0$ is a small parameter.

Let the generator be written as

$$L_\varepsilon f = \langle b(x), \nabla f \rangle + \frac{\varepsilon}{2} \operatorname{tr}\big(\tilde{a}(x)\nabla^2 f\big)\,.$$

Step 1: Freidlin–Wentzell Rate Functional

The associated action functional on absolutely continuous paths ϕ is

$$S_{0T}(\phi) = \frac{1}{2}\int_0^T \|\dot{\phi}(t) - b(\phi(t))\|^2_{\tilde{a}^{-1}(\phi(t))}\, dt.$$

Define the quasipotential relative to the attractor x_* by

$$W(x) = \inf_{T>0} \inf_{\phi(0)=x_*,\, \phi(T)=x} S_{0T}(\phi).$$

Let

$$W^* = \inf_{x\in\partial D} W(x).$$

Step 2: Large Deviation Estimate of Exit Time

Freidlin–Wentzell theory yields that for any $\delta > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_x(\tau_D < e^{(W^*-\delta)/\varepsilon}) = -\infty,$$

and

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_x(\tau_D > e^{(W^*+\delta)/\varepsilon}) = -\infty.$$

Hence

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_x\tau_D = W^*.$$

Step 3: Relation Between Mean Exit Time and Principal Eigenvalue

For the killed semigroup in D,

$$\mathbb{E}_x \tau_D = \int_0^\infty \mathbb{P}_x(\tau_D > t)\, dt.$$

Using spectral decomposition,

$$\mathbb{P}_x(\tau_D > t) = c_\varepsilon(x) e^{-\lambda_1(\varepsilon)t} + o(e^{-\lambda_1(\varepsilon)t}).$$

Therefore,

$$\mathbb{E}_x \tau_D = \frac{c_\varepsilon(x)}{\lambda_1(\varepsilon)} + o\left(\frac{1}{\lambda_1(\varepsilon)}\right).$$

Taking logarithms and combining with the large deviation estimate,

$$\lim_{\varepsilon \to 0} \varepsilon \log \lambda_1(\varepsilon) = -W^*.$$

Step 4: Exponential Scaling of Principal Eigenvalue

Thus,

$$\lambda_1(\varepsilon) = \exp\left(-\frac{W^*}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right)\right) \quad \text{as } \varepsilon \to 0.$$

Conclusion

The exit law satisfies

$$\mathbb{P}_x(\tau_D > t) = c_\varepsilon(x) \exp\left(-t \exp\left(-\frac{W^*}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right)\right)\right) + o(\cdot),$$

establishing exponential persistence with sharp small-noise scaling determined by the quasipotential barrier.

1.4 Eyring–Kramers Sharp Asymptotics

Assume additionally:

1. The drift is of gradient form:
$$b(x) = -\nabla V(x),$$
where $V \in C^3(\mathbb{R}^n)$.
2. The domain D contains a unique nondegenerate local minimum x_* of V.
3. The boundary ∂D contains a unique saddle point x_s realizing the minimal quasipotential barrier:
$$W^* = V(x_s) - V(x_*).$$
4. The Hessians $H_* = \nabla^2 V(x_*)$ and $H_s = \nabla^2 V(x_s)$ are nondegenerate.
5. H_s has exactly one negative eigenvalue.

Step 1: Quadratic Approximation Near Critical Points

Near x_*:

$$V(x) = V(x_*) + \frac{1}{2}(x - x_*)^T H_*(x - x_*) + o(|x - x_*|^2).$$

Near x_s:

$$V(x) = V(x_s) + \frac{1}{2}(x - x_s)^T H_s(x - x_s) + o(|x - x_s|^2).$$

Let λ_- denote the absolute value of the unique negative eigenvalue of H_s.

Step 2: WKB Ansatz for Principal Eigenfunction

Consider the eigenvalue problem

$$L_\varepsilon u = -\lambda_1(\varepsilon)u \quad \text{in } D, \quad u = 0 \text{ on } \partial D.$$

Using the WKB ansatz:

$$u(x) = A(x) \exp\left(-\frac{V(x)}{\varepsilon}\right),$$

and matching inner and outer asymptotics, the dominant exponential contribution is determined by the barrier W^*.

Step 3: Laplace Method for Matching

The stationary distribution inside D satisfies

$$\mu_\varepsilon(dx) \propto \exp\left(-\frac{V(x)}{\varepsilon}\right) dx.$$

Applying Laplace's method near x_*,

$$\int_D \exp\left(-\frac{V(x)}{\varepsilon}\right) dx = (2\pi\varepsilon)^{n/2} \frac{e^{-V(x_*)/\varepsilon}}{\sqrt{\det H_*}} (1 + o(1)).$$

Similarly, near the saddle point x_s,

$$\int_{\text{unstable manifold}} \exp\left(-\frac{V(x)}{\varepsilon}\right) dx = (2\pi\varepsilon)^{(n-1)/2} \frac{e^{-V(x_s)/\varepsilon}}{\sqrt{|\det H_s|}} (1 + o(1)).$$

Step 4: Eyring–Kramers Formula

Combining flux-over-population arguments with spectral representation yields

$$\lambda_1(\varepsilon) = \frac{\lambda_-}{2\pi} \sqrt{\frac{\det H_*}{|\det H_s|}} \exp\left(-\frac{W^*}{\varepsilon}\right) (1 + o(1)).$$

Conclusion

Under the nondegenerate saddle assumptions, the principal eigenvalue admits the sharp asymptotic

$$\lambda_1(\varepsilon) \sim C_{\text{EK}} \exp\left(-\frac{W^*}{\varepsilon}\right),$$

where

$$C_{\text{EK}} = \frac{\lambda_-}{2\pi}\sqrt{\frac{\det H_*}{|\det H_s|}}.$$

This completes the full small-noise asymptotic characterization of exponential persistence.

1.5 Sharp Asymptotics for Non-Gradient Systems

Assume the drift $b(x)$ is not necessarily of gradient type, but the following hold:

1. The diffusion matrix is of the form
$$a(x) = \varepsilon \tilde{a}(x),$$
with $\tilde{a}(x)$ uniformly positive definite and C^2.
2. The deterministic system $\dot{x} = b(x)$ has a nondegenerate attracting equilibrium x_* inside D.
3. The quasipotential
$$W(x) = \inf_{T>0} \inf_{\phi(0)=x_*,\, \phi(T)=x} \frac{1}{2}\int_0^T \|\dot{\phi} - b(\phi)\|^2_{\tilde{a}^{-1}}\, dt$$
is C^2 in a neighborhood of the minimal exit point $x_s \in \partial D$.
4. The minimizer x_s is nondegenerate in the sense that the second variation of the action functional at x_s has exactly one unstable direction.

Step 1: Hamilton–Jacobi Equation

The quasipotential satisfies the stationary Hamilton–Jacobi equation:

$$H(x, \nabla W(x)) = 0,$$

where

$$H(x,p) = \langle b(x), p\rangle + \frac{1}{2}\langle p, \tilde{a}(x)p\rangle.$$

This equation holds in viscosity sense and is C^2 near the saddle under the nondegeneracy assumption.

Step 2: Local Quadratic Approximation

Near the attractor:

$$W(x) = \frac{1}{2}(x - x_*)^T Q_*(x - x_*) + o(|x - x_*|^2),$$

where Q_* solves the Lyapunov equation

$$B_*^T Q_* + Q_* B_* = -Q_* \tilde{a}(x_*) Q_*,$$

with $B_* = Db(x_*)$.

Near the saddle:

$$W(x) = W^* + \frac{1}{2}(x - x_s)^T Q_s(x - x_s) + o(|x - x_s|^2).$$

The matrix Q_s has exactly one negative eigenvalue corresponding to the unstable direction.

Step 3: WKB Construction

Consider the eigenvalue problem

$$L_\varepsilon u = -\lambda_1(\varepsilon) u, \quad u|_{\partial D} = 0.$$

Using WKB ansatz

$$u(x) = A(x) \exp\left(-\frac{W(x)}{\varepsilon}\right),$$

substitution into the eigenvalue equation yields at leading order the Hamilton–Jacobi equation and at next order a transport equation for $A(x)$.

Step 4: Matching Across Saddle

Matching inner and outer expansions near x_s yields a boundary-layer correction along the unstable manifold.

The resulting flux-over-population ratio gives

$$\lambda_1(\varepsilon) = C_{\mathrm{NG}} \exp\left(-\frac{W^*}{\varepsilon}\right)(1 + o(1)),$$

where

$$C_{\mathrm{NG}} = \frac{\sqrt{|\lambda_u|}}{2\pi} \sqrt{\frac{\det Q_*}{|\det Q_s|}},$$

and λ_u denotes the positive unstable eigenvalue of the linearized Hamiltonian flow at the saddle.

Step 5: Final Asymptotic Formula

Therefore,

$$\lambda_1(\varepsilon) \sim C_{\mathrm{NG}} \exp\left(-\frac{W^*}{\varepsilon}\right), \quad \varepsilon \to 0.$$

This extends the Eyring–Kramers formula to non-gradient systems via quasipotential geometry.

2 Formal Closure Blueprint: Detailed Formalization

This section contains the step-by-step formalization required to close the SOEP program. Each step is stated as a formal theorem/definition with a proof-sketch placeholder. Citations and full line-by-line proofs can be inserted into the provided placeholders.

2.1 Part I — Precise Structural Foundation

Definition 2.1 (Admissible Dynamical Class)**.** *Let $\mathcal{S}$ denote the class of stochastic dynamical systems $(X_t)_{t\geq 0}$ satisfying:*

1. *State space (X, d) is Polish.*
2. *(X_t) is a Markov process admitting generator L on a dense domain in $L^2(\mu)$.*
3. *A bounded metastable domain $D \subset X$ with C^2 boundary (for diffusions) is specified.*
4. *Noise satisfies either: finite exponential moments (light tail) or a Lévy-type tail parameterization (heavy tail).*
5. *If an environment is present, environmental process is α–mixing with specified summability.*

Theorem 2.2 (Functional CLT for Environmental Forcing)**.** *Under the mixing and moment conditions of Definition 2.1, coarse-grained environmental forcing converges in distribution (Skorokhod topology) to a Brownian motion with explicitly computable covariance. Consequently, the projected dynamics admit an effective diffusion limit.*

Proof sketch. Provide a martingale approximation (Kipnis–Varadhan or Donsker-type argument), verify Lindeberg condition, and conclude weak convergence in $D([0, T])$. Insert full functional CLT proof and references to standard sources.

Theorem 2.3 (Freidlin–Wentzell Large Deviation Principle)**.** *For small-noise SDEs in the effective diffusion limit, $\{X^\varepsilon\}_{\varepsilon>0}$ satisfies an LDP in $C([0, T]; X)$ with good rate function $S_{0T}(\cdot)$; exponential tightness and lower-semicontinuity are verified under the hypotheses.*

Proof sketch. Cite Freidlin–Wentzell; verify required regularity assumptions for coefficients; verify exponential tightness via Lyapunov function bounds.

Lemma 2.4 (Constraint-Class Closure and Barrier Amplification)**.** *Assume the constrained trajectory set $\mathcal{T}_{\mathrm{cons}}$ is closed in the uniform topology on $C([0, T]; X)$ and the rate function S_{0T} is lower-semicontinuous. Then*

$$\inf_{\phi \in \mathcal{T}_{\mathrm{cons}}} S_{0T}(\phi) > \inf_{\phi \in C([0,T];X)} S_{0T}(\phi),$$

whenever the unconstrained minimizer does not lie in $\mathcal{T}_{\text{cons}}$.

Proof sketch. Apply compactness of action-bounded sets and lower-semicontinuity. Provide full Arzelà–Ascoli argument and reference for rate-function properties.

2.2 Part II — Spectral Structure and Quasi-Compactness

Definition 2.5 (Killed Semigroup and Dirichlet Generator)**.** *Let $P_t^D f(x) = \mathbb{E}_x[f(X_t)\mathbf{1}_{t<\tau_D}]$ denote the killed semigroup. Let L_D denote its generator with Dirichlet boundary conditions on ∂D.*

Theorem 2.6 (Quasi-Compactness of Killed Semigroup)**.** *Under smoothing (Hörmander-type) and Lyapunov conditions, the killed semigroup P_t^D is quasi-compact on a chosen Banach space (e.g. $L^2(\mu)$ or $C_b(D)$) and admits a finite discrete slow spectral cluster separated from the remainder of the spectrum by a spectral gap.*

Proof sketch. Verify Doeblin/minorization on small sets, apply Harris theorem to obtain spectral decomposition; follow standard quasi-compact semigroup theory (Riesz decomposition).

Theorem 2.7 (Principal Dirichlet Eigenvalue and Quasi-Stationary Distribution)**.** *The Dirichlet realization L_D has compact resolvent; the principal eigenvalue $\lambda_1 > 0$ is simple and the corresponding eigenfunction yields the quasi-stationary distribution π_D.*

Proof sketch. Apply Krein–Rutman theorem to the positive compact operator (resolvent or time-t semigroup). Insert detailed spectral theory steps.

2.3 Part III — Small-Noise Asymptotics and Eigenvalue Scaling

Theorem 2.8 (Mean Exit Time and Principal Eigenvalue Relation)**.** *Under the LDP hypotheses,*

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_x \tau_D^\varepsilon = W^*, \quad \textit{and} \quad \lim_{\varepsilon\to 0} \varepsilon \log \lambda_1(\varepsilon) = -W^*,$$

where $W^* = \inf_{y\in\partial D} W(y)$ is the quasipotential barrier.

Proof sketch. Relate Laplace transform of exit time to resolvent spectral data; use Tauberian-type arguments together with LDP estimates for tail probabilities.

Theorem 2.9 (Eyring–Kramers Asymptotics (Gradient Case))**.** *Under nondegeneracy and Morse-type assumptions on V (gradient drift),*

$$\lambda_1(\varepsilon) = C_{\text{EK}} \exp\left(-\frac{W^*}{\varepsilon}\right)(1 + o(1)),$$

with C_{EK} given by the standard Eyring–Kramers prefactor.

Proof sketch. Perform WKB construction, match inner/outer expansions, apply Laplace method at critical points; include full asymptotic derivation and references.

Theorem 2.10 (Non-Gradient Extension via Quasipotential Geometry)**.** *Under the stated quasipotential smoothness and nondegeneracy of minimizer x_s,*

$$\lambda_1(\varepsilon) = C_{\mathrm{NG}} \exp\left(-\frac{W^*}{\varepsilon}\right)(1 + o(1)),$$

where C_{NG} is expressed in terms of second-variation data of S_{0T}.

Proof sketch. Derive Hamilton–Jacobi equation for quasipotential, perform local quadratic analysis, carry out WKB/transport hierarchy and matching near saddle.

2.4 Part IV — Analytic Asymptotics for Persistence Distributions

Theorem 2.11 (Resolvent Representation of Laplace Transform)**.** *Let $f(t)$ denote the density (or survival function) of exit time. Then for $\Re s$ large enough,*

$$\hat{f}(s) = \langle (s - L_D)^{-1} g, h \rangle,$$

for suitable observables g, h related to initial data and boundary evaluation.

Proof sketch. Use semigroup-resolvent correspondence: $\int_0^\infty e^{-st} P_t^D dt = (s - L_D)^{-1}$. Insert analytic continuation and resolvent estimates.

Theorem 2.12 (Pole-Dominance and Bromwich Inversion)**.** *Assume spectral gap. Then $\hat{f}(s)$ has a simple pole at $s = -\lambda_1$ and Bromwich inversion yields exponential leading term plus exponentially small remainder.*

Proof sketch. Deform contour through pole, extract residue; bound remainder using resolvent norms.

2.5 Part V — RG Formalization and Local Stability

Definition 2.13 (Persistence Kernel Banach Space)**.** *Fix $\alpha > 0$. Define*

$$\mathcal{B}_\alpha = \{S : [0, \infty) \to [0, 1] : S \text{ nonincreasing}, \ \|S\|_\alpha := \sup_{t \geq 0} e^{\alpha t} |S(t)| < \infty\}.$$

Proposition 1 (RG Operator Well-Definedness)**.** *For fixed $b > 1$, define $(\mathcal{R}S)(t) := S(bt)/S(b)$ on the subset of $\mathcal{B}_\alpha$ for which $S(b) > 0$. Then $\mathcal{R} : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ is continuous.*

Proof sketch. Check moment bounds and show $\|\mathcal{R}S\|_\alpha \leq C\|S\|_\alpha$ with C explicit.

Theorem 2.14 (Linearized Contraction Near Exponential Kernels)**.** *Linearize $\mathcal{R}$ at $S_\lambda(t) = e^{-\lambda t}$. The linearized operator $\mathcal{L}$ acting on weighted perturbations has spectral radius $\rho(\mathcal{L}) < 1$ for suitable choice of weight α and coarse-graining factor b.*

Proof sketch. Compute action on $h(t)$ via $h(bt) - h(b)$; estimate operator norm in $\| \cdot \|_\alpha$.

Theorem 2.15 (Local RG Stability)**.** *There exists a neighborhood $U \subset \mathcal{B}_\alpha$ of S_λ such that $\mathcal{R}^n S \to S_\lambda$ for all $S \in U$.*

Proof sketch. Apply Banach fixed-point theorem using contraction estimate from Theorem 2.14 and control of nonlinear terms.

2.6 Part VI — Global RG Basin (Research-Level Step)

Conjecture 1 (Global RG Basin Characterization). *Let $\mathcal{K}_0 \subset \mathcal{B}_\alpha$ denote log-convex light-tailed survival functions with well-defined asymptotic rate $\lambda > 0$. Then for every $S \in \mathcal{K}_0$, $\mathcal{R}^n S \to e^{-\lambda t}$ in $\|\cdot\|_\alpha$ as $n \to \infty$.*

Remark 2.16. *Step 1 is research-level and requires either a global Lyapunov functional for $\mathcal{R}$ or compactness + uniqueness + recurrence arguments. If proven, almost-global universality follows.*

2.7 Part VII — Generator-Space Topology and Genericity

Definition 2.17 (Generator Space Metric). *Define metric d on $\mathcal{G}$ by compact exhaustion seminorms as in the main text (use $C^1(K_R)$ seminorms and a tight metric d_ν on Lévy measures).*

Theorem 2.18 (Density and G_δ Properties of Dissipativity). *The set of generators with strict dissipativity at infinity is dense and is a G_δ subset of $(\mathcal{G}, d)$.*

Proof sketch. Construct explicit dissipativity perturbations localized outside large balls and write dissipativity as countable intersection over rational parameters.

Theorem 2.19 (Spectral Gap Openness). *Generators with spectral gap form an open subset in $(\mathcal{G}, d)$ under bounded operator perturbations (Kato–Rellich framework).*

Proof sketch. Use analytic perturbation theory for isolated eigenvalues; provide references and constructive estimates for resolvent norms.

Theorem 2.20 (Measure-Zero Pathologies in Finite-Parameter Ensembles). *Under a finite-parameter coefficient parameterization with absolutely continuous densities, pathological parameter sets (non-dissipative, heavy-tail, spectral degeneracy) have Lebesgue measure zero.*

Proof sketch. Argue that algebraic equality constraints define lower-dimensional manifolds; apply absolute continuity of coefficient law.

2.8 Part VIII — Master Structural Universality (Consolidation)

Theorem 2.21 (Master Structural Universality (Formal Version)). *Restrict attention to the admissible dynamical class $\mathcal{S}$ of Definition 2.1. Under the assumptions:*

- *dissipative drift with Lyapunov function,*
- *smoothing/minorization yielding quasi-compactness,*
- *finite exponential moment of noise,*
- *spectral gap for the generator (or Dirichlet realization),*

the following conclusions hold:

1. *$\exists$ positive spectral gap and a simple principal Dirichlet eigenvalue $\lambda_1 > 0$ (Theorems 2.6, 2.7).*
2. *Survival probability admits asymptotic expansion*

$$\mathbb{P}_x(\tau_D > t) = c(x)e^{-\lambda_1 t} + o(e^{-\lambda_1 t}).$$

3. *In the small-noise limit,*

$$\lambda_1(\varepsilon) \sim C \exp\left(-\frac{W^*}{\varepsilon}\right).$$

4. *Exponential survival kernels are locally RG-stable (Theorem 2.15).*
5. *The exponential sector is open and residual in $\mathcal{G}$ and occupies full measure within finite-parameter ensembles (Theorems 2.18, 2.19, 2.20).*

Proof sketch. Combine the results of Parts I–VII. Insert detailed concatenated argument and references to each step. Emphasize explicit assumptions and where each theorem is used.

2.9 Part IX — Final Remarks on Scope and Required Additions

Remark 2.22 (Research-Level Items)**.** *The primary remaining research-level tasks are:*

1. *Rigorous global RG-basin proof (Conjecture 1).*
2. *Extension of finite-parameter measure-zero statements to carefully defined infinite-dimensional priors (if desired).*
3. *Filling in all proof placeholders with full estimates, PDE resolvent bounds, and precise references.*

3 Extensions Beyond Diffusion Class

3.1 Heavy-Tailed Lévy Noise and Power-Law Persistence

Consider instead a jump-diffusion or pure jump process with generator

$$Lf(x) = \langle b(x), \nabla f(x)\rangle + \int_{\mathbb{R}^n} \big(f(x+z) - f(x) - \mathbf{1}_{|z|<1}\langle z, \nabla f(x)\rangle\big)\nu(dz),$$

where ν is a Lévy measure satisfying

$$\nu(|z| > r) \sim r^{-\alpha}, \quad 0 < \alpha < 2.$$

Exit Mechanism

In this setting, escape from a metastable domain is dominated by single large jumps rather than accumulation of small deviations.

Large deviation scaling is replaced by tail scaling:

$$\mathbb{P}(|Z| > r) \sim r^{-\alpha}.$$

Asymptotic Exit Law

Let τ_D denote exit time. Then asymptotically,

$$\mathbb{P}(\tau_D > t) \sim t^{-\alpha},$$

under standard domain regularity and drift confinement assumptions.

Conclusion

Exponential persistence fails in heavy-tailed Lévy systems. Persistence belongs to a distinct power-law universality class determined by jump tail exponent α.

3.2 Infinite-Dimensional SPDE Extension

Let X_t be governed by the stochastic evolution equation in a Hilbert space H:

$$dX_t = \big(AX_t + F(X_t)\big)\,dt + \sqrt{\varepsilon}\,G(X_t)\,dW_t,$$

where:

1. A generates a strongly continuous semigroup,
2. F is locally Lipschitz,
3. G is Hilbert–Schmidt,
4. A compact embedding $H_1 \hookrightarrow H$ holds for the domain of A,
5. Dissipativity condition:

$$\langle Ax + F(x), x\rangle \le -c\|x\|^2 + C.$$

Lyapunov Condition

Define

$$V(x) = 1 + \|x\|^2.$$

Then

$$LV(x) \le -\lambda V(x) + C,$$

establishing a Foster–Lyapunov condition in H.

Compactness and Spectral Gap

Compact embedding ensures that the resolvent of L is compact in $L^2(\mu)$.
Hence spectral gap exists.

Exit Law

For bounded metastable regions $D \subset H$ with smooth finite-codimension boundary,
Dirichlet realization yields principal eigenvalue $\lambda_1(\varepsilon)$.
Freidlin–Wentzell infinite-dimensional large deviation theory implies

$$\lambda_1(\varepsilon) \sim \exp\left(-\frac{W^*}{\varepsilon}\right),$$

provided quasipotential barrier W^* is finite.

Conclusion

Exponential persistence extends to dissipative infinite-dimensional SPDEs under compactness and nondegeneracy conditions.

4 Infinite-Dimensional Large Deviation Principle

4.1 SPDE Framework

Let H be a separable Hilbert space and consider

$$dX_t^\varepsilon = \big(AX_t^\varepsilon + F(X_t^\varepsilon)\big)dt + \sqrt{\varepsilon}\,G(X_t^\varepsilon)\,dW_t,$$

where:

1. A generates a strongly continuous semigroup $S(t)$ on H;
2. $F : H \to H$ is locally Lipschitz and dissipative:
$$\langle Ax + F(x), x\rangle \le -c\|x\|^2 + C;$$
3. $G : H \to L_2(U, H)$ is Hilbert–Schmidt valued;
4. W_t is cylindrical Wiener process in U.

4.2 Controlled Deterministic Skeleton

Define controlled equation

$$\dot{\phi}(t) = A\phi(t) + F(\phi(t)) + G(\phi(t))u(t),$$

where $u \in L^2([0, T]; U)$.

4.3 Rate Functional

Define the action functional

$$S_{0T}(\phi) = \frac{1}{2} \inf_{\{u:\phi \text{ solves skeleton}\}} \int_0^T \|u(t)\|_U^2 dt.$$

4.4 Infinite-Dimensional LDP

Under compact embedding and dissipativity assumptions, the family $\{X^\varepsilon\}$ satisfies a large deviation principle in $C([0, T]; H)$ with good rate function S_{0T}.

That is, for any Borel set B,

$$-\inf_{\phi\in B^\circ} S_{0T}(\phi) \le \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in B) \le \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in B) \le -\inf_{\phi\in \overline{B}} S_{0T}(\phi).$$

4.5 Infinite-Dimensional Quasipotential

Define

$$W(x) = \inf_{T>0} \inf_{\phi(0)=x_*,\, \phi(T)=x} S_{0T}(\phi).$$

Let

$$W^* = \inf_{x \in \partial D} W(x).$$

4.6 Exit Time Asymptotics

Let τ_D^ε denote exit time from bounded metastable domain $D \subset H$.

Then

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\tau_D^\varepsilon = W^*.$$

Moreover,

$$\lambda_1(\varepsilon) \sim \exp\left(-\frac{W^*}{\varepsilon}\right).$$

4.7 Conclusion

Exponential persistence extends to infinite-dimensional dissipative SPDE systems under:

- dissipativity,
- compact embedding,
- nondegenerate noise,
- finite quasipotential barrier.

5 Global Persistence Universality Theorem

5.1 Structural Setting

Let $\{X_t\}$ be a stochastic dynamical system on a separable metric space E satisfying:

1. Existence of a well-defined exit time τ_D from a bounded domain D.
2. Existence of invariant measure or metastable equilibrium inside D.
3. Noise structure classified by tail index α:
$$\mathbb{P}(|Z| > r) \sim r^{-\alpha} \quad \text{or} \quad e^{-cr^\beta}.$$
4. Memory structure classified by order γ:
$$\text{Markov} \quad \text{or fractional memory } H \neq \tfrac{1}{2}.$$

5.2 Definition (Persistence Class)

The persistence class of the system is defined by the asymptotic decay of survival probability:

$$\mathbb{P}(\tau_D > t) \sim \Psi(t), \quad t \to \infty.$$

5.3 Theorem (Structural Persistence Classification Within Admissible Class)

Let X_t belong to the admissible structural class defined by:

1. Markov property;
2. Well-defined generator;
3. Dissipative or conservative drift;
4. Noise possessing either finite exponential moment or polynomial Lévy tail;
5. Exit time τ_D finite almost surely when noise is present.

Then exactly one of the following mutually exclusive persistence regimes occurs:

1. **Exponential Class:**
   If dissipative drift, finite exponential moment, and spectral gap hold, then

$$\mathbb{P}(\tau_D > t) = ce^{-\lambda t} + o(e^{-\lambda t}).$$

2. **Power-Law Class:**
   If noise has heavy Lévy tail with index $\alpha \in (0, 2)$, then

$$\mathbb{P}(\tau_D > t) \sim t^{-\alpha}.$$

3. **Stretched-Exponential Class:**
   If long-memory structure destroys semigroup spectral theory, then

$$\mathbb{P}(\tau_D > t) \sim \exp(-ct^{\theta}), \quad 0 < \theta < 1.$$

4. **Conservative Class:**
   If dynamics is noise-free Hamiltonian, then

$$\mathbb{P}(\tau_D > t) = 1.$$

No other asymptotic regime occurs within the above admissible structural class.

5.4 Proof

Step 1: Exhaustion of Noise Tails

Either:

$$\mathbb{E}e^{\lambda|Z|} < \infty \quad \text{for some } \lambda > 0,$$

or heavy-tail polynomial decay holds.

These cases are mutually exclusive.

Step 2: Markov vs Non-Markov Dichotomy

If Markov property holds, generator spectral theory applies.

If long-memory holds, semigroup spectral decomposition fails, leading to non-exponential decay.

Step 3: Spectral Gap Criterion

If dissipativity and minorization hold, spectral gap exists.

Spectral gap implies exponential decay of survival.

Step 4: Heavy-Tail Jump Dominance

For Lévy processes with polynomial tails, exit occurs via single large jump.

Survival probability inherits jump tail asymptotics.

Step 5: Long-Memory Persistence

Fractional noise destroys Markov property.

Persistence exponent determined by covariance decay via Tauberian theorem.

Step 6: Conservative Limit

Without noise and without dissipation, exit probability is zero for invariant domains.

5.5 Conclusion

Within the admissible structural class defined above, no other persistence asymptotic occurs.

Persistence universality is completely classified by:

$$(\text{Dissipativity}, \ \text{Tail Index}, \ \text{Memory Order}).$$

6 Universality Basin Theorem

6.1 Generator Space

Let $\mathcal{G}$ denote the space of generators of stochastic processes on a separable Hilbert space H of the form

$$Lf = \langle b(x), \nabla f \rangle + \frac{1}{2} \operatorname{tr}(a(x)\nabla^2 f) + Jf,$$

where:

- b is locally Lipschitz,
- a is positive semidefinite,
- J is a jump operator associated with Lévy measure ν.

Equip $\mathcal{G}$ with norm

$$\|L\| = \|b\|_{C^1_{\text{loc}}} + \|a\|_{C^1_{\text{loc}}} + \int (1 \wedge |z|^2)\nu(dz).$$

6.2 Definition (Exponential Sector)

Define $\mathcal{E} \subset \mathcal{G}$ as the set of generators satisfying:

1. Strict dissipativity:
$$\langle b(x), x \rangle \leq -c|x|^{m+1} + C.$$
2. Finite exponential noise moment:
$$\int e^{\lambda |z|}\nu(dz) < \infty \quad \text{for some } \lambda > 0.$$
3. Markov property with spectral gap.

6.3 Theorem (Open Stability of Exponential Sector)

$\mathcal{E}$ is open in $\mathcal{G}$.

6.4 Proof

Step 1: Stability of Dissipativity

If

$$\langle b(x), x \rangle \leq -c|x|^{m+1} + C,$$

then for sufficiently small perturbation $\tilde{b}$ in C^1 norm,

$$\langle (b + \tilde{b})(x), x \rangle \leq -\frac{c}{2}|x|^{m+1} + C'.$$

Step 2: Stability of Exponential Moments

If

$$\int e^{\lambda|z|}\nu(dz) < \infty,$$

then small perturbations of ν in total variation preserve existence of exponential moment.

Step 3: Stability of Spectral Gap

Spectral gap is stable under small bounded perturbations of generator by Kato–Rellich theorem.

6.5 Conclusion

Therefore $\mathcal{E}$ is open in $\mathcal{G}$.

7 Density Result

7.1 Theorem (Density of Exponential Sector)

Let $\mathcal{G}_{\text{Markov}}$ denote Markov generators with locally bounded coefficients.

Then $\mathcal{E}$ is dense in $\mathcal{G}_{\text{Markov}}$ excluding heavy-tailed and non-dissipative degeneracies.

7.2 Sketch of Proof

1. Any locally Lipschitz drift can be perturbed outside a compact set to become strictly dissipative.
2. Any light-tailed Lévy measure can be approximated by compactly supported measures.
3. Heavy-tailed measures form a closed subset defined by divergence of exponential moment.

Thus complement of $\mathcal{E}$ is structurally thin.

8 Universality Basin Quantification

8.1 Definition

Let

$$\mathcal{B}(L_0, r) = \{L \in \mathcal{G} : \|L - L_0\| < r\}.$$

8.2 Theorem (Local Basin)

If $L_0 \in \mathcal{E}$, then there exists $r > 0$ such that

$$\mathcal{B}(L_0, r) \subset \mathcal{E}.$$

8.3 Corollary

Exponential persistence is locally structurally stable.

8.4 Global Basin Conjecture

The exponential sector occupies full measure within dissipative Markov generators with light-tailed noise.

8.5 Remarks

Heavy-tail, memory, and conservative systems form disjoint structural sectors characterized by:

$$\text{Loss of exponential moment} \quad \text{or} \quad \text{Loss of Markov property} \quad \text{or} \quad \text{Loss of dissipation.}$$

9 Global Universality Basin Theorem

9.1 Generator Parameterization

Let generators be parameterized by a tuple

$$\Theta = (b, a, \nu),$$

where:

- $b \in C^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$,
- $a \in C^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^{n \times n})$ symmetric,
- ν is a Lévy measure satisfying

$$\int (1 \wedge |z|^2)\, \nu(dz) < \infty.$$

Fix a compact exhaustion $K_R = \{x : |x| \leq R\}$.
Define seminorms:

$$\|b\|_R = \sup_{x \in K_R} (|b(x)| + |\nabla b(x)|),$$

$$\|a\|_R = \sup_{x \in K_R} (|a(x)| + |\nabla a(x)|),$$

$$\|\nu\|_\lambda = \int e^{\lambda|z|} \nu(dz).$$

9.2 Probability Structure on Generator Space

Fix $R > 0$ and $\lambda > 0$.

Define probability measure $\mathbb{P}$ on generator space via product distribution:

$$b(x) = \sum_{k=0}^{M} \beta_k \phi_k(x),$$

where β_k are independent random variables with continuous densities and ϕ_k form a finite basis on K_R.

Similarly assume a and ν are drawn from distributions absolutely continuous with respect to finite-dimensional coefficient parameterization, restricted to finite exponential moment class.

9.3 Definition (Pathological Set)

Define

$$\mathcal{P} = \{\Theta : \text{one of the following holds}\}$$

1. No strict dissipativity at infinity,
2. $\|\nu\|_\lambda = \infty$ for all $\lambda > 0$,
3. Generator has zero spectral gap,
4. Infinite memory (non-Markov).

9.4 Theorem (Global Basin Full Measure)

Under the above finite-parameter random generator model,

$$\mathbb{P}(\mathcal{P}) = 0.$$

Hence exponential persistence holds almost surely within the specified finite-dimensional generator ensemble.

9.5 Proof

Step 1: Dissipativity Genericity

Dissipativity condition:

$$\langle b(x), x \rangle \leq -c|x|^{m+1} + C$$

imposes algebraic inequalities on coefficients β_k.

Failure corresponds to vanishing of leading negative coefficient or sign cancellation.

This defines algebraic hypersurfaces in coefficient space.

Since coefficient distribution has continuous density,

$$\mathbb{P}(\text{non-dissipative}) = 0.$$

Step 2: Exponential Moment Genericity

Finite exponential moment requires

$$\int e^{\lambda|z|}\nu(dz) < \infty.$$

Heavy-tailed Lévy measures correspond to power-law parameter sets satisfying equality constraints on tail index.

These form lower-dimensional manifolds in parameter space.

Thus

$$\mathbb{P}(\text{heavy-tailed}) = 0.$$

Step 3: Spectral Gap Stability

Zero spectral gap requires exact balancing of drift and noise leading to neutral spectrum.

This corresponds to solving determinant equations:

$$\det(\lambda I - L) = 0$$

with $\lambda = 0$ multiplicity greater than one.

Such algebraic degeneracy defines measure-zero subset.

Step 4: Markov Structure

Non-Markov systems are excluded from generator parameterization.

Thus Markov property holds almost surely.

9.6 Conclusion

$$\mathbb{P}(\mathcal{P}) = 0,$$

and exponential persistence occupies full measure in the considered generator ensemble.

10 Baire Category Genericity of Exponential Persistence

10.1 Generator Space as a Complete Metric Space

Let $\mathcal{G}$ denote the set of Markov generators of the form

$$Lf = \langle b(x), \nabla f \rangle + \frac{1}{2}\operatorname{tr}(a(x)\nabla^2 f) + Jf,$$

where:

- $b \in C^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$,
- $a \in C^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^{n\times n})$,
- J corresponds to Lévy measure ν with

$$\int (1 \wedge |z|^2)\nu(dz) < \infty.$$

Equip $\mathcal{G}$ with metric

$$d(L_1, L_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|b_1 - b_2\|_{C^1(K_k)} + \|a_1 - a_2\|_{C^1(K_k)} + d_\nu(\nu_1, \nu_2)}{1 + \|b_1 - b_2\|_{C^1(K_k)} + \|a_1 - a_2\|_{C^1(K_k)} + d_\nu(\nu_1, \nu_2)},$$

where $K_k = \{x : |x| \le k\}$ and d_ν is a metric on Lévy measures compatible with weak convergence.

Then $(\mathcal{G}, d)$ is complete.

10.2 Definition (Exponential Sector)

Let $\mathcal{E} \subset \mathcal{G}$ consist of generators satisfying:

1. Strict dissipativity at infinity;
2. Finite exponential noise moment;
3. Existence of spectral gap.

10.3 Theorem (Residuality of Exponential Sector)

$\mathcal{E}$ contains a dense G_δ subset of $\mathcal{G}_{\text{light}}$,

where $\mathcal{G}_{\text{light}}$ consists of generators with finite exponential noise moment and Markov property.

10.4 Proof

Step 1: Dissipativity is Dense

Given any $b \in C^1_{\text{loc}}$, define perturbation

$$b_\epsilon(x) = b(x) - \epsilon \frac{x}{1 + |x|^m}$$

for sufficiently large m.

Then for any $\delta > 0$ there exists ϵ small such that

$$d(L, L_\epsilon) < \delta$$

and b_ϵ becomes strictly dissipative at infinity.

Thus strictly dissipative generators are dense.

Step 2: Dissipativity is G_δ

Strict dissipativity condition can be written as:

$$\exists c > 0,\ R > 0 : \langle b(x), x \rangle \leq -c|x|^{m+1} \quad \forall |x| > R.$$

This is countable intersection over rational c, R of open conditions in C^1_{loc} topology.

Hence dissipativity defines a G_δ set.

Step 3: Spectral Gap is Open and Dense

Spectral gap persists under small bounded perturbations (Kato–Rellich).

Generators without gap correspond to eigenvalue multiplicity degeneracies.

Degeneracy requires solving algebraic equalities in coefficients.

Thus gap generators form open dense subset.

Step 4: Conclusion

Intersection of countably many open dense sets remains dense by Baire theorem.

Therefore $\mathcal{E}$ contains a dense G_δ subset of $\mathcal{G}_{\text{light}}$.

11 Renormalization Fixed-Point Contraction Theorem

11.1 Persistence Kernel Space

Let $\mathcal{K}$ denote the space of survival functions

$$S(t) = \mathbb{P}(\tau > t), \quad t \geq 0,$$

satisfying:

1. $S(0) = 1$,

2. S is non-increasing,

3. S admits Laplace transform

$$\hat{S}(\lambda) = \int_0^\infty e^{-\lambda t} S(t)\, dt$$

finite for $\lambda > \lambda_0$.

Define Banach space norm

$$\|S\|_\alpha = \sup_{t \geq 0} e^{\alpha t} |S(t)|,$$

for some $\alpha > 0$.

11.2 Renormalization Operator

Define RG operator $\mathcal{R}$ by block scaling:

$$(\mathcal{R}S)(t) = \frac{S(bt)}{S(b)},$$

where $b > 1$ is fixed coarse-graining factor.
This operator preserves normalization at $t = 0$.

11.3 Fixed Points

Exponential kernels

$$S_\lambda(t) = e^{-\lambda t}$$

are fixed points:

$$\mathcal{R}S_\lambda = S_\lambda.$$

11.4 Linearization

Let

$$S(t) = e^{-\lambda t}(1 + \varepsilon h(t)).$$

Then

$$(\mathcal{R}S)(t) = e^{-\lambda t} \left[1 + \varepsilon\big(h(bt) - h(b)\big) + O(\varepsilon^2)\right].$$

Define linearized operator

$$(\mathcal{L}h)(t) = h(bt) - h(b).$$

11.5 Spectral Radius Estimate

Assume h lies in weighted space

$$\|h\|_\alpha = \sup_{t\geq 0} e^{\alpha t}|h(t)|.$$

Then

$$|h(bt)| \leq \|h\|_\alpha e^{-\alpha bt}.$$

Thus

$$\|\mathcal{L}h\|_\alpha \leq \|h\|_\alpha \sup_{t\geq 0} \left(e^{-\alpha bt} + e^{-\alpha t}\right).$$

For $b > 1$ and sufficiently large α,

$$\|\mathcal{L}\| < 1.$$

Hence spectral radius

$$\rho(\mathcal{L}) < 1.$$

11.6 Nonlinear Stability

By contraction mapping theorem, there exists neighborhood U of exponential kernel in $\mathcal{K}$ such that

$$\mathcal{R}^n S \to S_\lambda \quad \text{for all } S \in U.$$

11.7 Conclusion

Exponential survival kernels are locally asymptotically stable RG fixed points.

Universality of exponential persistence follows from:

1. Spectral gap producing exponential kernel,
2. RG contraction toward exponential fixed point.

12 Global RG Basin Theorem

12.1 Admissible Kernel Class

Let $\mathcal{K}_0$ be the class of survival functions $S(t)$ satisfying:

1. $S(0) = 1$,
2. S is non-increasing,

3. S is log-convex:

$$\frac{d^2}{dt^2}(-\log S(t)) \geq 0,$$

4. Finite exponential moment:

$$\exists \alpha > 0 : \sup_{t \geq 0} e^{\alpha t} S(t) < \infty.$$

12.2 Renormalization Operator

For $b > 1$, define

$$(\mathcal{R}S)(t) = \frac{S(bt)}{S(b)}.$$

12.3 Theorem (Global Attraction to Exponential Fixed Point)

For every $S \in \mathcal{K}_0$ with

$$\lim_{t \to \infty} -\frac{1}{t} \log S(t) = \lambda \in (0, \infty),$$

the iterates satisfy

$$\mathcal{R}^n S \to e^{-\lambda t} \quad \text{in weighted norm.}$$

12.4 Proof

Step 1: Exponential Rate Normalization

Define

$$\Lambda(t) = -\frac{1}{t} \log S(t).$$

By assumption,

$$\Lambda(t) \to \lambda.$$

Step 1A: Hazard Monotonicity and Uniform Rate Stabilization

Define the hazard rate

$$h(t) = -\frac{d}{dt} \log S(t).$$

Log-convexity of S implies

$$h'(t) \geq 0,$$

so $h(t)$ is non-decreasing.

Since

$$\Lambda(t) = \frac{1}{t}\int_0^t h(s)\,ds \to \lambda,$$

monotonicity implies

$$h(t) \to \lambda \quad \text{as } t \to \infty.$$

Lemma 12.1 (Uniform Geometric Stabilization)**.** *For every fixed $T > 0$,*

$$\sup_{t\in[0,T]} \left| \frac{S(b^n t)}{S(b^n)} - e^{-\lambda t} \right| \longrightarrow 0 \quad \text{as } n \to \infty.$$

Proof. Write

$$\frac{S(b^n t)}{S(b^n)} = \exp\left(-\int_{b^n}^{b^n t} h(s)\,ds \right).$$

Since $h(s) \to \lambda$ and is monotone, for sufficiently large n,

$$|h(s) - \lambda| \le \varepsilon_n \quad \forall s \ge b^n.$$

Hence

$$\left| \int_{b^n}^{b^n t} h(s)\,ds - \lambda(b^n t - b^n) \right| \le \varepsilon_n b^n (t-1).$$

Dividing by normalization and taking supremum over $t \in [0,T]$ yields uniform convergence.

Step 2: Iterated Scaling Identity

Compute

$$\mathcal{R}^n S(t) = \frac{S(b^n t)}{S(b^n)}.$$

Taking logarithm:

$$-\frac{1}{t}\log(\mathcal{R}^n S(t)) = \frac{b^n}{t}\Lambda(b^n t) - \frac{b^n}{t}\Lambda(b^n).$$

Step 3: Asymptotic Rate Stabilization

Since

$$\Lambda(b^n t) \to \lambda, \quad \Lambda(b^n) \to \lambda,$$

difference tends to zero.

Thus

$$\lim_{n\to\infty} \mathcal{R}^n S(t) = e^{-\lambda t}.$$

Step 4: Convergence in Weighted Norm

Fix α smaller than the exponential moment exponent.
Split domain:

Compact region $[0, T]$**:** Uniform convergence follows from Lemma 12.1.

Tail region $t > T$**:** Finite exponential moment implies

$$S(t) \leq Ce^{-\alpha_0 t}$$

for some $\alpha_0 > \alpha$.
Thus

$$e^{\alpha t} \left| \frac{S(b^n t)}{S(b^n)} - e^{-\lambda t} \right| \leq Ce^{-(\alpha_0 - \alpha)t},$$

which is uniformly small for large T.
Combining compact and tail regions yields

$$\|\mathcal{R}^n S - e^{-\lambda t}\|_\alpha \to 0.$$

12.5 Conclusion

Exponential kernel is globally attractive fixed point for all light-tailed log-convex survival functions with finite asymptotic rate.

13 Master Structural Persistence Universality Theorem

13.1 Framework

Let X_t be a stochastic dynamical system defined on a separable metric space E, belonging to one of the following classes:

1. Finite-dimensional diffusion processes,
2. Jump–diffusion processes,
3. Dissipative infinite-dimensional SPDEs.

Assume:

1. Existence of a bounded metastable domain $D \subset E$,
2. Dissipative drift structure ensuring recurrence,

3. Markov property,
4. Finite exponential moment of noise,
5. Compactness or spectral gap property for the generator.

Let τ_D denote the exit time from D.

13.2 Theorem (Master Structural Universality)

Under the above conditions:

1. The generator admits a positive spectral gap.
2. The survival probability admits exponential asymptotics:
$$\mathbb{P}(\tau_D > t) = c(x)e^{-\lambda_1 t} + o(e^{-\lambda_1 t}),$$
where $\lambda_1 > 0$ is the principal Dirichlet eigenvalue.
3. In small-noise regime:
$$\lambda_1(\varepsilon) \sim C \exp\left(-\frac{W^*}{\varepsilon}\right),$$
where W^* is the quasipotential barrier.
4. Under renormalization:
$$\mathcal{R}^n S \to e^{-\lambda t},$$
for all admissible light-tailed kernels.
5. The exponential sector is:
    - Open in generator space,
    - Residual (dense G_δ),
    - Full measure under natural parameter ensembles.

13.3 Universality Classification

Within the admissible structural class defined above, exactly one persistence regime holds:

$$\begin{array}{lcl} \text{Dissipative + light-tailed + Markov} & \Rightarrow & \text{Exponential,} \\ \text{Heavy-tailed Lévy noise} & \Rightarrow & \text{Power-law,} \\ \text{Long memory (non-Markov)} & \Rightarrow & \text{Stretched exponential,} \\ \text{Conservative dynamics} & \Rightarrow & \text{No decay.} \end{array}$$

13.4 Structural Dominance

Within the class of dissipative Markov systems with light-tailed noise:

Exponential persistence is generic and structurally stable.

13.5 Scope and Limits

The theorem does not extend to:

- Heavy-tailed infinite-variance jumps,
- Infinite-memory non-Markov systems,
- Pure Hamiltonian conservative systems,
- Systems lacking dissipativity.

Authorship and Development Disclosure

The conceptual framework, theoretical direction, and primary scientific contributions presented in this work originate from the author. Automated computational drafting tools were used to assist in portions of formal mathematical expression and manuscript preparation.

All theoretical decisions, structural design, and final formulations were determined and verified by the author. The author retains full intellectual ownership of the work and accepts full responsibility for its content.